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SIMULTANEOUS DESIGN OF COMMUNICATION STRATEGIES AND CONTROL POLICIES IN STOCHASTIC SYSTEMS

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SIMULTANEOUS DESIGN OF COMMUNICATION STRATEGIES AND CONTROL POLICIES IN STOCHASTIC SYSTEMS

BY

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TABLE OF CONTENTS

CHAPTER 1

INTRODUCTION

1.1. Motivation

In many engineering situations a variety of possible measurements can be carried out on a physical system or process, and the question that naturally arises is what measurement process is optimal. Thus it may be necessary to communicate information through noisy channels to an observer, and a decision on what information to transmit may need to be made. Based on the observations received, the controller is required to make decisions that may affect the cost incurred as well as the further evolution of the system.

Consider, for example, the following decentralized systems:

(a) *Control of a Space Probe from an Earth Station:* We may think of a space probe and an earth station together *constituting a decentralized system*. The sensor aboard the space probe is required to transmit data over the downlink, after suitable encoding at some permissible power level. The earth station may be required to command the space probe to move along a certain trajectory, or it may wish to form estimates of the transmitted measurement. The problem, then, is one of simultaneously designing the communication strategy used by the probe and the control policy of the earth station.

(b) *Tracking Systems:* In tracking systems it may be possible to allow for transmitting measurements at larger power levels (with an increase in the associated transmission cost) in order to combat noise and increase control efficiency. Here not only the communication strategy but also the transmission power level need to be designed.

Problems requiring simultaneous communication and control also arise in many socioeconomic situations, control of flexible structures, and other areas where the design

or allocation of limited transmission resources is important. The essentials of the situations involve

(i) A stochastic system, usually dynamic and/or decentralized, consisting of a number of decision makers (synonymously agents) along with a mathematical description of their interaction with the system and among themselves.

(ii) Elements of uncertainty (noises) entering the system, the underlying probability spaces for which are beyond the control of the decision makers.

(iii) An information structure, which characterizes the information gained and recalled by each decision maker.

(iv) A set of possible alternatives (decisions) for each agent, and permissible strategies which are mappings from the information space to the decision space of each agent.

(v) An *objective* functional that summarizes the preference ordering among various alternatives for the set of decision makers.

We note here that for the class of problems described above there exist very general formulations where, for example, the order in which the agents act may be determined by a chance mechanism (Witsenhausen [1971]) or the various agents exhibit a divergence of interests requiring a different objective functional for each agent (Başar and Olsder [1982]). Here we restrict our attention to classes of problems where in which

(i) The order in which the agents act is fixed in advance.

(ii) All noises which may enter the system are zero mean Gaussian, and further

(iii) The problem involves the design of two kinds of strategies:

- Communication strategies for generating information-bearing signals.

- Control policies for forming estimates, minimizing errors and reducing costs.

Many such problems have been the subject of previous investigations. Athans [1972] considers the problem of selecting, at each instant of time, one measurement provided by one out of many sensors, with each measurement having an associated cost. For the class of linear stochastic dynamic systems and measurement subsystems, a weighted combination of prediction cost and accumulated observation cost is minimized, with the optimal measurement strategy being obtained by solving a deterministic optimal control problem. A similar formulation was considered earlier by Meier et al. [1967] who obtained a computational solution using dynamic programming. Herring and Melsa [1974] generalize the above results to allow the selection, at each instant of time, of the best combination of measurement devices as opposed to the best single device, and Mehra [1976] presents measurement schedules and sensor designs for linear stochastic systems subject to a constraint on the total measurement precision so as to minimize a norm of the error covariance matrix.

In contrast to the above studies, where the cost is the covariance of the error of the Kalman estimate, Chu [1978] considers the problem of finding the best measurement for static problems with arbitrary quadratic cost. A similar framework to that of Chu is adopted by Papavassilopoulos [1983] for the case of more than one decision maker, with the measurements still being restricted to be linear in the state.

Lafortune [1985] presents a general theorem for the computation of optimal solution to discrete time stochastic control problems when the decision makers have the additional freedom of choosing at each step among different sets of observations. Lafortune applies his results to finite state, controlled Markov chains as well as linear Gaussian systems with quadratic cost functionals. The control of Markov chains is also considered by

Varaiya and Walrand [1983], who obtain a complete solution for a class of symmetric channels with noiseless feedback.

Whittle and Rudge [1976] consider a situation where one agent observes a stationary ergodic Gaussian process and on its basis forms a sequence, through not necessarily a linear transformation. The other agent observes a noisy replica of the first agent's decision, and he is required to form an estimate of the original Gaussian sequence. This problem of simultaneous communication and control is solved using the concepts of statistical communication theory, and the solution requires that arbitrarily large signal blocks be available at all stages. In control theoretic terms their solutions are unrealizable, since the actions taken at a given time depend also on observations which lie in the future.

The problem considered by Whittle and Rudge is somewhat different from the ones mentioned earlier, because in the earlier formulations the measurement model is specified except for certain parameters, and the optimal selection of these parameters has been investigated. Here, we shall formulate and solve a class of problems related to that of Whittle and Rudge under the additional restriction of causality. When the design of the measurement strategy itself is part of the problem, the problem becomes considerably more difficult, since the action of one decision maker affects the information of the other, and there is no way in which this other decision maker can have access to the information on which the first one acted. Such information structures have been called *nonclassical* in the stochastic team literature. In the next section we study different information structures and the related computational considerations.

1.2. Information Structures and Computational Considerations

In this section we discuss the various types of information structures associated with stochastic team problems. These information structures characterize the precise static or dynamic information gained or recalled by each decision maker at each stage of a decision process, and are customarily distinguished as classical, quasiclassical or nonclassical. We then discuss some of the issues of computational complexity which arise when the numerical derivation of the optimal team solution is considered.

1.2.1. Classical and quasiclassical information structures

Classical information patterns include deterministic patterns and centralized information patterns. Deterministic patterns arise when the information is not noise-corrupted and may be of the *open-loop* type in which only the initial value of the state is available and no dynamic information is acquired, or of the *closed-loop* type where perfect information concerning the current value of the state is also acquired. Centralized patterns arise when all agents exchange their measurements without any delay and also recall the past information.

Under the deterministic or centralized stochastic information patterns, stochastic team problems become equivalent to stochastic control problems and the solution techniques for these (e.g., Bertsekas [1987], Kumar and Varaiya [1986]) are directly applicable. Thus for stochastic teams with classical information patterns, when

(i) the measurements are linear in the primitive random variables and controls,

(ii) the primitive random variables are jointly Gaussian,

(iii) the cost function is quadratic in the control vector and the primitive random variables, and

1.2.2. Nonclassical information structures

An information pattern is *nonclassical* if it is not partially nested. Alternatively, if the decision maker j's action affects the information of i, and there is no way by which i can infer the information available to j, then the information structure is said to be of the nonclassical type. Under the nonclassical information pattern, the derivation of the optimal team solution meets with formidable difficulties. One way of viewing these difficulties is that the control plays a triple role (Ho [1980]), viz.,

(i) the deterministic control effort of reducing the error;

(ii) improvement of future knowledge of uncertainty;

(iii) signalling to agents acting in the future some useful information that they do not necessarily acquire;

and these roles are in general conflicting.

Classes of tractable problems with nonclassical information have been very difficult to identify. Witsenhausen [1968] established that some of the simplest linear-quadratic-Gaussian (LQG) stochastic teams with nonclassical information do not admit optimal linear solutions; in fact the optimal solution to the problem formulated by Witsenhausen is not yet known. Very recently, however, some success has been reported for two-person stochastic teams with nonclassical information. Bansal and Başar [1987a] show that it is the presence of the product term between the decision variables which, coupled with the nonclassical information, makes the LQG problem intractable, and in the absence of this product term optimal solutions may readily be found. Some nonclassical multipath systems have also been shown to admit optimal linear solutions under quadratic costs and Gaussian noises (Bansal and Başar [1987b]).

Here we are concerned with nonclassical information patterns because many problems involving the simultaneous design of communication and control strategies exhibit an information pattern of this type. Recall that the third role of controlling agents in nonclassical teams mentioned above is the role of "signalling" information to agents acting in the future, and this role is absent in the case of classical and quasiclassical information structures. When explicit design of the communication strategy itself is considered, it is the "signalling" policy that is being designed, and nonclassical patterns arise naturally.

It is for the above reasons that the problem considered by Whittle and Rudge [1976], where the communication strategy is to be designed, is significantly more complicated than the other problems, where the measurement model is specified, except for certain parameters, as in Athans [1972], Herring and Melsa [1974], Mehra [1976].

In the course of this work we shall identify some classes of stochastic teams with nonclassical information which are tractable in spite of the above mentioned difficulties.

1.2.3. Computational complexity of stochastic team problems

The severe difficulty encountered when the solution of stochastic team problems is attempted via numerical techniques has triggered some research into their complexity. Some recent results (Papadimitriou and Tsitsiklis [1982], Tsitsiklis and Athans [1985]) relate the complexity of stochastic team problems to that of known intractable problems. Using the tools of computational complexity the NP completeness of the discrete version of the static team problem has been established, and some progress has also been made in defining complexity concepts for continuous time problems by relating the complexity of the continuous version to that of its discretized counterpart (Papadimitriou and Tsitsiklis [1986]). In the same work the NP completeness of the discrete version of Witsenhausen's

problem has also been established, thus explaining the failures, reported in literature, to attack it computationally (Ho and Chang [1980]).

Quite recently, Witsenhausen [1988] has developed a notion of equivalence between stochastic control problems that is suitable for complexity analysis. He has shown that a large class of problems with dynamic information can be reduced to equivalent static problems with a transformed cost functional, and this class includes all sequential discrete variable problems. Person-by-person optimality is then a sufficient condition for global optimality under smoothness and convexity conditions; however, when there is no convexity then person-by-person optimality is not sufficient.

While approaches based on the theory of NP completeness shed important light on the applicability (or nonapplicability) of numerical techniques for discretized versions of stochastic teams, being a worst case scenario they obscure important special cases which may admit relatively simple optimal solutions. For example, both the static team problem as well as Witsenhausen's counterexample have discretized versions which are NP complete, but the former is quite tractable under linear quadratic Gaussian assumptions (Radner [1962]), whereas the latter is still unsolved. The problem of carving out classes of tractable problems within the general class of nonclassical stochastic teams is an important one, and this thesis is a contribution to this direction.

1.3. Nonclassical Patterns and the Problem of Information Transmission

Consider the simplest problem of transmitting information, where the value of a random variable is to be transmitted, after suitable encoding, over a noisy channel. The noisy channel output is available at a decoder where it is used to construct an estimate of the input variable. In order to make this problem well defined, it is customary in the

communications literature to assume that the encoder operates under a hard power constraint, i.e., the mean-square level at the output of the encoder cannot exceed a certain predefined level.

This standard information transmission problem is depicted in Figure 1.1, where x is the input to be transmitted, u is the encoder output satisfying $E[u^2] \leq P^2$, w is the channel noise, and the output of the channel

$$y = u + w$$

is used by the decoder to form an estimate of x, designated by $\hat{x}$.

Note that the communication problem is one with a nonclassical information structure, since the action of the encoder affects the information of the decoder, but the decoder does not have access to the information of the encoder. (Indeed, if the decoder did have access to the information of the encoder, then the problem would be trivial, the estimate of x exact, and the channel redundant.)

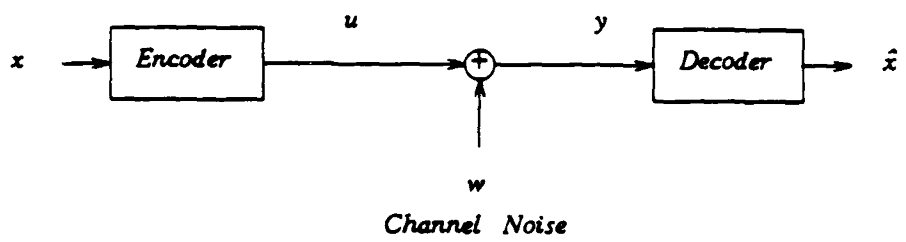

Figure 1.1. The information transmission problem.

Information Theory provides the fundamental results for the analysis of the information transmission system of Figure 1.1. One of the most important results of Information Theory relates channel capacity to the rate distortion of the source. (A study of

channel capacity and rate distortion may be found, for example, in Ash [1965], Berger [1971] and Gallager [1968].)

Let R(D) denote the rate distortion function, (Berger [1971]), corresponding to a source under an appropriate distortion measure D. Shannon [1959] proved that it is not possible with any coding scheme to transmit the source under consideration through any channel of capacity less than R(D) without incurring an average distortion larger than or equal to D. Conversely, given any channel of capacity $C > R(D)$, coding schemes exist (possibly using arbitrarily large block lengths) which result in an average distortion arbitrarily close to D when used over this channel. Thus, R(D) is the minimum channel capacity required to reproduce the given source at the decoder with average distortion at most D. For any channel, the minimum possible distortion D^* that can result from its use is related to its capacity C by

$$R(D^*) = C .$$

We shall find the above relation between capacity and rate distortion quite useful for the analysis of some decentralized stochastic team problems, because the information transmission problem is a special case of problems with nonclassical information. We will also need the concepts of entropy and conditional entropy (see, for example, Abramson [1963]); these will be introduced in subsequent chapters as required.

1.4. Organization of the Thesis

The organization of this thesis is as follows. In Chapter 2 we formulate and analyze some fundamental classes of stochastic team problems with two decentralized agents, exhibiting a nonclassical information pattern. We identify instances of such problems for which the optimal solutions are linear and may readily be found despite the intractability

of the general problem.

In Chapter 3 we consider a stochastic dynamic decision problem, where at each step two consecutive decisions must be made, one being what information-bearing signal to transmit, and the other what control action to exert. Such problems arise in the simultaneous optimization of both the observation and the control sequences in stochastic systems. This problem is solved completely for first-order auto-regressive moving average (ARMA) systems under the quadratic cost criterion. The results are further extended to cases where the time horizon is infinite and the cost function is discounted.

In Chapter 4 we consider stochastic dynamic decision problems requiring simultaneous optimization of both the observation and the control sequences for second- and higher-order systems, under quadratic cost criteria, and find strategies which are optimal over the affine class.

In Chapter 5 we generalize the results on the decentralized two-person teams of Chapter 2, by allowing the action of one agent to be transmitted to the other agent through a number of noisy channels simultaneously, instead of being transmitted through a single noisy channel. In Chapter 6 the results of Chapter 2 are extended in another direction, viz., to the case of more than two decision makers.

In Chapter 7 we expand on the framework by allowing an incomplete statistical description of the channel used to transmit measurements between the decentralized agents, and seek optimal solutions under a worst case analysis. It is assumed that the unknown part of the channel noise is controlled by an adversary or "jammer," and the situation is viewed as a zero sum game. The problems are studied for a variety of fidelity criteria, under hard and soft power constraints, for the transmitter as well as the jammer.

Finally, Chapter 8 provides a recapitulation of the results obtained, and concludes the thesis.

CHAPTER 2

THE DECENTRALIZED TWO-PERSON TEAM

2.1. Introduction

In this chapter we formulate and analyze some fundamental classes of stochastic team problems involving two decentralized agents, where the action of one agent affects the information of the other, and the other agent does not have access to the information upon which the first one acted. In Section 2.2 we formulate the general two-person stochastic team problem with quadratic costs and Gaussian noises. In Section 2.3 we identify those instances of the general problem for which the optimal solutions are linear. In Section 2.4 we show that for some instances in which we cannot show that linear solutions are globally optimal, the optimal linear policy may be outperformed by an appropriate nonlinear strategy. In Section 2.5 we comment on some aspects of the difficulties associated with Witsenhausen's counterexample, (Witsenhausen [1968]), and the concluding remarks in Section 2.6 end this chapter.

2.2. Problem Formulation

In this section, we formulate a two-person, decentralized, stochastic team problem with nonclassical information, where the action of one agent affects the information of the other and the information structure is nonnested (Basar and Cruz [1982]). The problem is to design the controls u_0 and u_1 so as to minimize the quadratic cost $J(\gamma_0,\gamma_1)$, where

$$J(\gamma_0,\gamma_1) = E[k_0u_0^2 + s_0u_0x + su_1^2 + s_1u_1x] ,$$

with

$$u_0 = \gamma_0(z)$$

and

$$u_1 = \gamma_1(y),$$

$k_0 > 0$, $s > 0$, s_0 and s_1 being prespecified constants.

The variables z and y, upon which the two agents base their respective actions, are given by

$$z = ax + v \tag{2.1}$$

and

$$y = u_0 + bx + w. \tag{2.2}$$

Here x, v and w are zero mean Gaussian random variables, which are independent of one another, and have variances σ_x^2, σ_v^2 and σ_w^2, respectively.

The situation is depicted schematically in Figure 2.1 below.

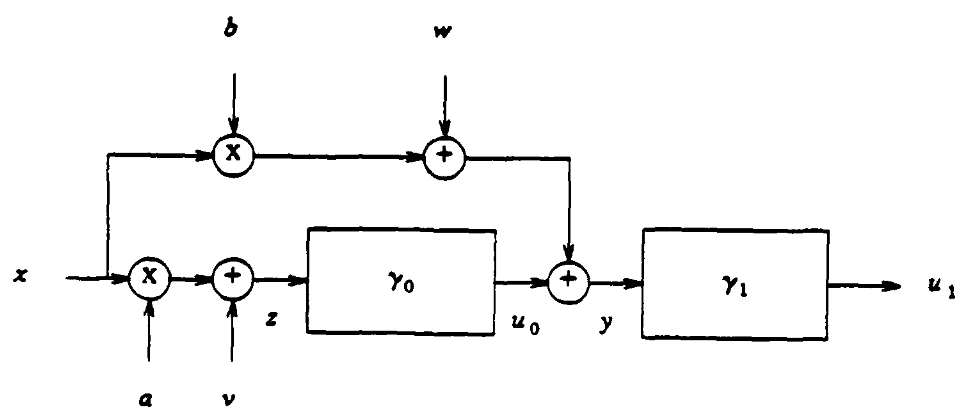

Figure 2.1. The two-person, decentralized, stochastic team.

There are no general analytical or numerical tools which can be used to obtain the optimal solution of this problem, the difficulty arising mainly due to its nonclassical information structure. The separation principle (Wonham [1968]), for example, does not apply here,

since the estimation part cannot be separated from the control action. Numerical techniques fail to provide any answers because the discretized version is NP complete (Tsitsiklis and Athans [1985]). To further understand the nature of the underlying difficulty, we may view the above *team* problem as an equivalent *control* (one-person) problem, with the control exhibiting a dual role. We complete the squares to obtain the cost functional

$$E[k_0u_0^2 + s_0u_0x + s(u_1 + \frac{s_1}{2s}x)^2],$$

which implies that the optimal γ_1 is necessarily given by

$$u_1 = \gamma_1(y) = -\frac{s_1}{2s}E(x \mid y) \tag{2.3}$$

and hence we have the equivalent problem:

$$\underset{\gamma_0}{Minimize}\; E[k_0(u_0 + \frac{s_0}{2k_0}x)^2 - \frac{s_1^2}{4s}E(E(x \mid y)^2)]$$

subject to (2.2).

We thus have a quadratic term to be minimized as is usual in stochastic control formulations; in addition, there is a second *nonquadratic* term which is influenced by the information about x that y contains. In the absence of this second term the minimum is attained by choosing u_0 as a linear function of z:

$$u_0 = \gamma_0(z) = -\frac{s_0}{2k_0}E(x \mid z) = -\frac{s_0}{2k_0}\left[\frac{a\sigma_x^2}{(a^2\sigma_x^2+\sigma_v^2)}\right]z, \tag{2.4}$$

but the presence of this second term brings in the possibility of a conflict between the two roles of control and information transmission. Thus, even if affine laws are optimal,

indirect techniques need to be used to prove their optimality; but in fact, as to be elucidated below, affine strategies do not always continue to be optimal.

2.3. Instances with Optimal Linear Solutions

In this section we show that if either $b=0$ or $\sigma_v^2=0$, then the stochastic team problem formulated in the preceding section admits optimal solutions which are linear in the observation variables.

(a) $\sigma_v^2=0$, b arbitrary

If $\sigma_v^2=0$, then we have $a=1$ without loss of generality, and we define $u'_0=u_0+bx$ to obtain the equivalent Problem P1 below:

$$\underset{\gamma'_0,\gamma'_1}{\text{Minimize}}\ J'(\gamma'_0,\gamma'_1)$$

where

$$J'(\gamma'_0,\gamma'_1)=E[k_0u'^2_0+s'_0u'_0x+\frac{s_1^2}{4s}(u'_1-x)^2+K]$$

$$u'_0=\gamma'_0(x)\,, \tag{2.5}$$

$$u'_1=\gamma'_1(u'_0+w)\,, \tag{2.6}$$

$$s'_0\equiv s_0-2k_0b\,. \tag{2.7}$$

and K is a constant independent of u'_0 and u'_1.

We thus obtain a problem of the type studied in Bansal and Başar [1987a], without a product term between the decision variables, for which linear strategies have been shown to be optimal. We briefly outline the approach here for completeness. Note that the situation is as depicted in Figure 2.2. In view of the discussion in Section 1.3, in which prob-

lems with nonclassical information have been related to problems involving information transmission, the agent taking action u_0 may be viewed as a generalized encoder and the agent taking action u_1 as a decoder, in the terminology of the information transmission system introduced in Chapter 1.

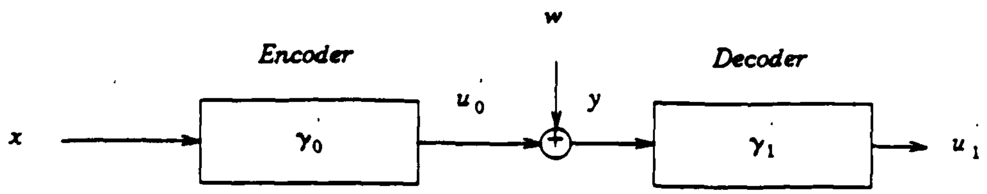

Figure 2.2. The transformed Problem P1.

We first consider the transformed problem under the additional restriction

$$E[u_0'^2] \leqslant P^2 . \tag{2.8}$$

Note that

$$I(x;y) \geqslant I(x;u'_1) \geqslant \frac{1}{2} \log \frac{\sigma_x^2}{E[(u'_1 - x)^2]} \tag{2.9}$$

where I(a;b) denotes the mutual information of random variables a and b. The first inequality in (2.9) is the data processing inequality, and the second inequality follows from the definition of mutual information (see, for example, Wyner [1970]).

Further, we have

$$I(x;y) = H(y) - H(y \mid x) \tag{2.10}$$

where H(y) is the entropy of the random variable y and H(y|x) is the conditional entropy of y given x.

Using (2.10) along with the fact that for a given variance the entropy is maximized by a Gaussian random variable with that variance, (see, for example, Kagan, Linnik and Rao [1973]), we have

$$I(x;y) \leqslant \frac{1}{2}\log\left(\frac{P^2+\sigma_w^2}{\sigma_w^2}\right). \tag{2.11}$$

Using (2.11) along with (2.9) we obtain (under the restriction (2.8))

$$E[(u'_1 - x)^2] \geqslant \frac{\sigma_x^2\sigma_w^2}{P^2+\sigma_w^2}. \tag{2.12}$$

Now let J'_p denote the minimum of $J'(\gamma'_0,\gamma_1)$ under the hard power constraint, i.e.,

$$J'_p \triangleq \inf_{u'_0,u'_1,E[u'^2_0]=P^2} J'(\gamma'_0,\gamma'_1). \tag{2.13}$$

We then have

$$\begin{aligned}
J'_p &\geqslant k_0P^2 + \operatorname*{Inf}_{E[u'_0]=P^2} E[s'_0u'_0x] + \operatorname*{Inf}_{E[u'^2_0]=P^2} E[(u'_1 - x)^2]\frac{s_1^2}{4s} + K \\
&\geqslant k_0P^2 + \operatorname*{Inf}_{E[u'_0]\leqslant P^2} E[s'_0u'_0x] + \operatorname*{Inf}_{E[u'^2_0]\leqslant P^2} E[(u'_1 - x)^2]\frac{s_1^2}{4s} + K \\
&= k_0P^2 - |s'_0|P\sigma_x + \frac{\sigma_x^2\sigma_w^2}{(P^2+\sigma_w^2)}\frac{s_1^2}{4s} + K \\
&\geqslant \operatorname*{Min}_{P\geqslant 0}\left[k_0P^2 - |s'_0|P\sigma_x + \frac{\sigma_x^2\sigma_w^2}{(P^2+\sigma_w^2)}\frac{s_1^2}{4s}\right] + K \\
&= k_0P^{*2} - |s'_0|P^*\sigma_x + \frac{\sigma_x^2\sigma_w^2}{(P^2+\sigma_w^2)}\frac{s_1^2}{4s} + K .
\end{aligned} \tag{2.14}$$

We thus obtain a lower bound on the optimal cost, and the final task is to note that this lower bound is tight and is achieved by using the linear policy:

$$u^*_0 = -(\operatorname{sgn} s'_0)\frac{p^*}{\sigma_x}\,x\,. \tag{2.15}$$

We now have Theorem 2.1 below:

Theorem 2.1.

(i) The stochastic team problem of Section 2.2 with $\sigma_v^2 = 0$, admits an optimal solution which is linear in the observation variables, and is given by

$$u^*_0 = \gamma^*_0(x) = p^*x \tag{2.16}$$

$$u^*_1 = \gamma^*_1(y) = -\frac{s_1}{2s}\left[\frac{(p^*+b)\sigma_x^2}{(p^*+b)^2\sigma_x^2+\sigma_w^2}\right]y \tag{2.17}$$

where

$$p^* = \lambda^* - b$$

and λ^* is given by the solution to the following parameter optimization problem:

$$\lambda^* = \arg\min_{\lambda}[k_0\lambda^2\sigma_x^2 + s'_0\lambda\sigma_x^2 + \frac{s_1^2}{4s}\frac{\sigma_x^2\sigma_w^2}{(\lambda^2\sigma_x^2+\sigma_w^2)}]\,. \tag{2.18}$$

(ii) The optimal value of the cost is

$$k_0\lambda^{*2}\sigma_x^2 + s'_0\lambda^*\sigma_x^2 + \frac{s_1^2}{4s}\frac{\sigma_x^2\sigma_w^2}{(\lambda^{*2}\sigma_x^2+\sigma_w^2)} + K \tag{2.19}$$

where

$$K = \sigma_x^2(-k_0b^2 + s_0b - s_1^2/4s)\,.$$

Remark 2.1. The parameter optimization problem (2.18) always admits a solution since as $\lambda \to \infty$, so does the expression to be minimized, implying that we can restrict the search to a compact set over which a continuous function always admits a minimum.

(b) $b = 0$, σ_v^2 arbitrary

If $b = 0$, then the channel noise becomes independent of the input x, and the problem is a special case of the multipath system considered in Bansal and Başar [1987b] for which linear strategies have been shown to be optimal. For completeness we outline the approach here. We can assume, without any loss of generality, that $a = 1$, then introduce the random variable

$$m = E(x \mid z) \tag{2.20}$$

and make the following observations:

(i) $\gamma_1(y) = -\dfrac{s_1}{2s} E(x \mid y)$

(ii) $E[E(x \mid y) - x)^2] = E[(E(x \mid y) - m)^2] + E[(m - x)^2]$

(iii) $E[u_0(z)x] = E[u_0(z)m]$

(since the random variable $(x-E(x|z))$ is independent of z).

(iv) The random variable m is Gaussian distributed, since

$$m = \frac{\sigma_x^2}{\sigma_x^2+\sigma_v^2} z,$$

and has the variance

$$\sigma_m^2 = \sigma_x^4/(\sigma_x^2 + \sigma_v^2).$$

In view of the above observations, the problem with $b = 0$ is equivalent to

$$\underset{\gamma'_0,\gamma_1}{\text{Minimize}}\ J''(\gamma'_0,\gamma_1)$$

where

$$J''(u'_0,u_1) = E[k_0 {u'_0}^2 + s_0 u'_0 m + \frac{s_1^2}{4s}(u_1 - m)^2 + K']$$

with

$$u'_0 = \gamma'_0(m)$$
$$u_1 = \gamma_1(y)$$

and K' a constant independent of u'_0 and u_1. Since m is zero-mean Gaussian, we have a problem of the type discussed in part (a) of this section, and the optimality of linear strategies follows.

Theorem 2.2.

(*i*) The stochastic team problem of Section 2.2 with $b = 0$, $a = 1$, admits an optimal solution which is linear in the observation variables, and is given by

$$u^*_0(z) = \lambda^* z$$
$$u^*_1(y) = -\frac{s_1}{2s}\left[\frac{\lambda^* \sigma_x^2}{\lambda^{*2}(\sigma_x^2+\sigma_v^2)+\sigma_w^2}\right] y$$

where λ^* is given by the solution to the following parameter optimization problem:

$$\lambda^* = \arg\min_{\lambda}\left[k_0\lambda^2(\sigma_x^2 + \sigma_v^2) + s_0\lambda\sigma_x^2 + \frac{s_1^2}{4s}\left[\frac{\sigma_x^2(\lambda^2\sigma_v^2+\sigma_w^2)}{\lambda^2(\sigma_x^2+\sigma_v^2)+\sigma_w^2}\right]\right]$$

(a solution to which may always be found as discussed in Remark (2.1)).

(ii) The optimal value of the cost is

$$k_0\lambda^{*2}(\sigma_x^2+\sigma_v^2)+s_0\lambda^*\sigma_x^2+\frac{s_1^2}{4s}\frac{\sigma_x^2(\lambda^{*2}\sigma_v^2+\sigma_w^2)}{(\lambda^{*2}(\sigma_x^2+\sigma_v^2)+\sigma_w^2)}-\frac{s_1^2}{4s}\sigma_x^2 .$$

□

2.4. Nonoptimality of Linear Strategies

In the preceding section we have identified those instances of the general two-person, decentralized, stochastic team problem for which the optimal strategies are linear and may be obtained through the solution of a related parameter optimization problem. These include the cases where either $\sigma_v^2=0$, i.e., an uncorrupted version of the variable x is available to the agent acting first; or $b=0$, i.e., the channel noise is independent of the input x. In this section we show that when neither of the above two conditions hold, then it is possible to construct some problem instances where the optimal linear strategy is out-performed by an appropriately chosen nonlinear strategy. In the following, we first assume that $b=1$, $\sigma_w^2=0$. Note that if these assumptions imply that linear strategies are not optimal, then we cannot expect linear strategies to be optimal, in general, for the class of two-person decentralized teams with nonzero b and σ_v^2.

Problem PG

$$\underset{\gamma_0,\gamma_1}{\text{Minimize}}\ E[k_0u_0^2+s_0u_0x+su_1^2+s_1u_1x]$$

where

$$u_0=\gamma_0(z)\,,\quad u_1=\gamma_1(y)$$

Note that, since $z=ax+v$, x may be written as $x=\beta z+n$ where

$$\beta=a\sigma_x^2/(a^2\sigma_x^2+\sigma_v^2)$$

and n is a Gaussian random variable which is independent of z, with variance

$$\sigma_n^2 = \sigma_x^2 \sigma_v^2 / (a^2 \sigma_x^2 + \sigma_v^2) .$$

Thus we obtain the equivalent problem

$$\underset{\gamma_0, \gamma_1}{\text{Minimize}}\ E[k_0 u_0^2 + \beta s_0 u_0 z + s u_1^2 + s_1 \beta u_1 z + s_1 u_1 n]$$

where

$$u_0 = \gamma_0(z) , \quad u_1 = \gamma_1(u_0 + \beta z + n) .$$

(In this restatement we have used the fact that $E[u_0 n] = 0$, n being independent of z, and having zero-mean.) We next define $u_0 + \beta z = u'_0$ to obtain the cost functional

$$E[k_0(u'_0 - \beta z)^2 + \beta s_0(u'_0 - \beta z)z + s u_1^2 + s_1 \beta u_1 z + s_1 u_1 n]$$

and arrive at the problem below:

$$\underset{\gamma'_0, \gamma_1}{\text{Minimize}}\ E[k_0 u'^2_0 + (\beta s_0 - 2k_0\beta) u'_0 z + s u_1^2 + s_1 u_1 n + s_1 \beta u_1 z]$$

where

$$u'_0 = \gamma'_0(z) , \quad u_1 = \gamma_1(u'_0 + n) \equiv \gamma_1(y) .$$

We thus have the situation depicted in Figure 2.3 below.

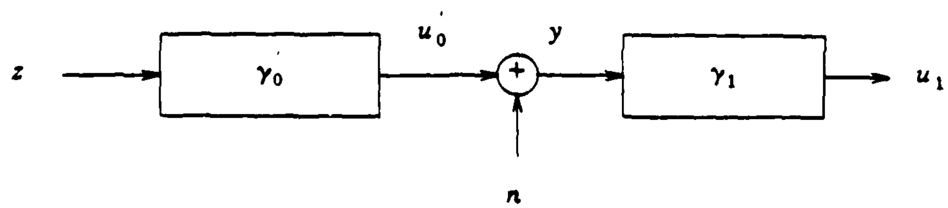

Figure 2.3. Schematics for the transformed problem.

Note the presence of the term $E[u_1 n]$ in the cost functional; in the absence of this term the

optimality of linear strategies may be established via information theoretic bounds. We now have a problem which is similar to the Gaussian test channel problem (Gallager [1968]), with the difference that instead of having to estimate the input z, the decoder is required to estimate a linear combination of the input and the channel noise. We thus have the problem below:

$$\underset{\gamma'_0,\gamma_1}{\text{Minimize}}\; E[k_0 u'^2_0 + s'_0 u'_0 z + s(u_1 + s_{11} n + s_{12} z)^2]$$

where

$$\begin{aligned} u'_0 &= \gamma'_0(z)\ ,\ \ u_1 = \gamma_1(u'_0 + n) \\ s'_0 &\equiv \beta(s_0 - 2k_0) \\ s_{11} &\equiv s_1/2s \\ s_{12} &\equiv s_1\beta/2s\,. \end{aligned}$$

We now note that since $n = y - u'_0$, we have

$$\begin{aligned} & u_1 + s_{11} n + s_{12} z \\ &= u_1 + s_{11}(y - u'_0) + s_{12} z \\ &= u_1 + s_{11} y + s_{12} z - s_{11} u'_0\,. \end{aligned}$$

Defining

$$u'_1 = u_1 + s_{11} y$$

we obtain the problem

$$\underset{\gamma'_0,\gamma'_1}{\text{Minimize}}\; E[k_0 u'^2_0 + s'_0 u'_0 z + s(u'_1 + s_{12} z - s_{11} u'_0)^2]$$

where

$$u'_0 = \gamma'_0(z)\,,\ \ u'_1 = \gamma'_1(y)\,.$$

This is a problem of the type studied in Bansal and Basar [1987a] with a product term between the decision variables, and hence, as shown there, regions of the parameter space

exist where nonlinear strategies outperform the optimum linear strategies.

An Illustration

Consider the Problem PG with parameter values $k_0 = 0.01$, $s_0 = 6.02$, $s = 1.0$, $s_1 = 2.0$, $\sigma_x^2 = 1.0025$, $a = -0.5/(1.0025)$ and $\sigma_v^2 = 100-(0.5)^2/1.0025$.

With the above choice we may write $x = \beta z + n$ where $\beta = -0.005$ and $\sigma_n^2 = 1.0$. (Recall that we have taken $\sigma_w^2 = 0$, $b = 1$ and $\sigma_z^2 = 100.0$).

We then obtain the transformed problem

$$\underset{\gamma'_0,\gamma'_1}{\text{Minimize}}\ E[0.01u'^2_0 - 0.03u'_0x + (u'_1 - u'_0 - 0.005z)^2]$$

where

$$u'_0 = \gamma'_0(z)$$
$$u'_1 = \gamma'_1(u'_0 + n) \equiv \gamma'_1(y)\,.$$

Now, if γ'_0 is linear, then optimal γ'_1 is also linear, and hence restricting γ'_0 to the form $\gamma'_0 = \lambda z$, and optimizing over λ, yields a cost of -1.2477 which is the best in the linear class. However, we can show that the nonlinear policy

$$\gamma'_0(z) = 10 \text{ sgn } z$$

$$\gamma'_1(y) = \begin{cases} 10 & \text{if } y \geqslant 0 \\ -10 & \text{if } y < 0 \end{cases}$$

yields a cost of -1.3911 which is superior to the optimal cost in the linear class.

2.5. Some Comments on Witsenhausen's Counterexample

The following problem WI, studied by Witsenhausen [1968], is a two-person, decentralized, stochastic team problem with a product term between the decision variables, and may be viewed as a special case of the reduced version found in Section 2.4.

Problem WI

$$\underset{\gamma_0,\gamma_1}{\text{Minimize}}\ E[k_0(u_0 - x)^2 + (u_0 - u_1)^2]$$

where

$$u_0 = \gamma_0(x)\ ;\ u_1 = \gamma_1(y)$$

and

$$y = u_0 + w\ .$$

Noting that

$$(u_0 - u_1)^2 = (y - w - u_1)^2$$

and defining

$$u'_1 = y - u_1\ ,$$

we obtain the equivalent Problem WI' below.

Problem WI'

$$\underset{\gamma_0,\gamma'_1}{\text{Minimize}}\ E[k_0(u_0 - x)^2 + (u'_1 - w)^2]$$

with

$$u_0 = \gamma_0(x)\,;\, u'_1 = \gamma'_1(u_0 + w)\,.$$

We, therefore, have a problem in which the second agent wishes to estimate the channel *noise* (which is independent of u_0), in sharp contrast to the communication type problems where the second agent wishes to estimate the observation of the first agent.

Since the second agent necessarily uses $u'_1 = E(w \mid y)$, the second term in the cost function for Problem WI′ becomes

$$J_2(\gamma_0) = E[(w - E(w \mid y))^2]\,.$$

It is a known result in probability and statistics that for every fixed power level $E[u_0^2] = P^2$, the linear strategy

$$u_0 = \gamma_0(x) = \frac{P}{\sigma_x}\, x$$

maximizes $J_2(\gamma_0)$, when $y = u_0 + w$. To prove this result, consider a zero sum game in normal form with kernel

$$G(u_0,u_1) = E[(w - u_1)^2]$$

which is to be maximized by a choice of $u_0 = \gamma_0(x)$ and minimized by a choice of $u_1 = \gamma_1(y)$, subject to the constraint $E[u_0^2] = P^2$. Note that

$$\operatorname*{Max}_{u_0} J_2(u_0) = \operatorname*{Max}_{u_0} \operatorname*{Min}_{u_1} G(u_0,u_1)\,,$$

i.e., the unrestricted maximum of the function J_2 is equal to the lower value of the game with kernel G. But the upper and lower value of a game are equal if a saddle point exists, and to complete the proof one can show that

$$u_0 \sim N(0,P^2)$$

$$u_1 = \frac{\sigma_w^2}{P^2+\sigma_w^2}\, y$$

provide a saddle-point solution for the game above.

In view of the above result, we see that for a given power level the linear choice of strategy γ_0 is the *worst possible* design as far as the minimization of the second term in the cost functional for Problem WI′ is concerned. It is therefore not surprising that linear solutions are not optimal for the classes of problems which involve the estimation of channel noise as in the case of Witsenhausen's counterexample, as well as in the general case of Section 2.4.

2.6. Conclusion

In this chapter we have formulated and analyzed some fundamental classes of stochastic team problems, involving two decentralized agents, where the action of one agent affects the information of the other, and the information structure is nonnested. We have identified those instances of the general problem for which the optimal solutions are linear. For some instances in which we cannot show that linear strategies are globally optimal, we have shown that the optimal linear policies may be outperformed by appropriately chosen nonlinear strategies. We have commented on some aspects of the difficulty associated with Witsenhausen's problem. The important conclusion that can be drawn from the analysis in this chapter is that for two-person decentralized stochastic team problems there do exist classes of parameter values for which the optimal solutions are linear and may readily be found, despite the fact that the general problem is quite intractable.

CHAPTER 3

SIMULTANEOUS COMMUNICATION AND CONTROL: FIRST-ORDER ARMA MODELS WITH FEEDBACK

3.1. Introduction

In this chapter we consider stochastic dynamic teams where at each step two consecutive decisions must be taken, one being what information-bearing signal to transmit and the other regarding what control action to exert. The organization of this chapter is as follows. In Section 3.2 we shall formally pose the problem with hard power constraints as Problem P1. In Section 3.3 we shall use an intermediate Problem P2 to construct Problem P3. Problems P1 and P3 will be shown to be equivalent in the sense that the optimal solution of one may be constructed from the optimal solution of the other. In Section 3.4 we formulate and solve an auxiliary problem using some results from Information Theory. In Section 3.5 we shall show that the solution to the auxiliary problem may be used to provide a solution to Problem P3, thereby solving Problem P1. In Section 3.6 we study the associated "soft" constraint version, and in Section 3.7 we study the existence of stationary optimal policies for the infinite horizon problem with discounted cost. The concluding remarks in Section 3.8 then end this chapter.

3.2. Problem Formulation

In this section we give a precise formulation of the problem to be solved in the sequel.

The problem is to control a stochastic dynamic system, the description of which is available in the form of a general first-order difference equation, with the objective being to minimize a quadratic cost functional.

The control v_i is based on the observation vector $(y_0,\ldots,y_i)$, where each of the y_k's is a noise-corrupted version of the information-bearing signal u_i, the u_i's being based on the current value of the state and all previous y_k's. In the context of the space probe example of Section 1.1, (with the u_i's being generated on board the space probe and the v_i's being generated on the earth station), such a situation arises quite naturally because a feedback line (the uplink) is available which is many times more reliable than the forward link (the downlink).

The problem clearly has a nonclassical information structure, since the agent taking action v_i does not have access to the information based on which action u_i is taken. A precise formulation of this nonclassical stochastic team problem is provided next.

We specify the stochastic system by the following set of equations:

$$x_{i+1} = \rho_i x_i + m_i - v_i \tag{3.1a}$$

$$y_i = u_i + w_i \tag{3.1b}$$

along with

$$E[u_i^2] \leqslant P_i^2 \tag{3.1c}$$

$$u_i = h_i(x_i, y^{i-1}) \tag{3.1d}$$

and

$$v_i = \gamma_i(y^i) \ . \tag{3.1e}$$

Here (3.1a) and (3.1b) are the state and measurement equations, respectively. The subscript $i,(i=0,1,\ldots)$ denotes that the realization of the random variable is at the i-th time instant, whereas the superscript denotes the history of the random variable up to that time instant, i.e.,

$$y^i = (y_0, \ldots, y_i) \ .$$

which we are also going to interpret as a row vector. The random variables x_o, w_i and m_i ($i \geq 0$) are all assumed to be independent and Gaussian with mean zero and variance $\sigma^2_{(\cdot)}$, the subscript being the identifier. The functions h_i and γ_i, $i=0,1,\ldots$, are the control policies, each one Borel-measurable in its arguments, and leading to second-order random variables u_i and v_i, respectively, the former also satisfying the power constraint (3.1c).

The criterion for comparing different control policies h_i's and γ_i's is based on the cost function we wish to optimize, which in this case is taken to be

$$J(h^N,\gamma^N) = E[\sum_{i=0}^{N} (a_{i+1}x_{i+1}^2 + b_i v_i^2)] \tag{3.2}$$

where $a_{i+1}>0$ and $b_i>0$ for all $i=0,\ldots,N$, and the action variables u_i and v_i are related to the policy variables h_i and γ_i via (3.1d)-(3.1e). We thus have Problem P1 below.

Problem P1:

$$\underset{h^N,\gamma^N}{\text{Minimize}} \quad J(h^N,\gamma^N)$$

subject to (3.1a) through (3.1e), where $J(h^N,\gamma^N)$ is defined by (3.2).

3.3. Construction of an Equivalent Problem

The gist of this section is as follows: First an equivalent Problem P2 is constructed from P1 which differs only in the form of the cost function. The cost function for P2 is in the form of a sum of the squared differences between state and control variables. In the transformation from P1 to P2 the constraints represented by (3.1a) through (3.1e) are unaltered. In a follow-up step Problem P3 is constructed from Problem P2 such that the

structure of the cost function is left unaltered, while the state equations are redefined so as to facilitate subsequent analysis.

These two transformations are presented below as Claims 3.1 and 3.2, respectively.

Claim 3.1: Under the set of constraints represented by (3.1a) through (3.1e), the cost function for Problem P1, defined by (3.2), is identical to

$$J''(n^N,\gamma^N) = E[\sum_{i=0}^{N} a'_i(v_i-b'_ix_i)^2] + c_N \tag{3.3}$$

where

$$b'_i \triangleq k_{i+1}\rho_i/(b_i+k_{i+1}) \tag{3.4a}$$

$$a'_i \triangleq b_i+k_{i+1} \tag{3.4b}$$

$$c_N \triangleq k_o\sigma^2_{x_o} + \sum_{i=0}^{N} k_{i+1}\sigma^2_{m_i} \tag{3.4c}$$

and $\{k_i\}$ is a sequence defined recursively by

$$k_i = a_i + k_{i+1}b_i\rho_i^2/(b_i+k_{i+1}) \tag{3.5}$$

$$k_{N+1} = a_{N+1}$$

Proof: This is a standard result in stochastic LQ control with perfect state information (see, e.g., Bertsekas [1987] or Kumar and Varaiya [1986]), known also as "completing the squares." Note that c_N is a constant (independent of the control sequence $\{v_i\}$), and (3.5) is the so-called discrete-time Riccati equation for this scalar problem.

□

Claim 3.2: The solution to Problem P2 may be obtained by solving the following equivalent Problem P3.

Problem P3:

$$\underset{h^N,\gamma^N}{\text{Minimize}} \qquad J(h^N,\gamma^N)$$

where

$$\tilde{x}_{i+1} = \rho_i\tilde{x}_i + m_i \tag{3.6a}$$

$$\tilde{v}_i = \gamma_i(y^i) \tag{3.6b}$$

$$y_i = u_i + w_i \tag{3.6c}$$

$$u_i = h_i(\tilde{x}_i, y^{i-1}) \tag{3.6d}$$

and the u_i's satisfy

$$E[u_i^2] \leqslant P_i^2 \tag{3.6e}$$

with

$$J(h^N,\gamma^N) = E[\sum_{i=0}^{N} a'_i(\tilde{v}_i - b'_i\tilde{x}_i)^2] + c_N \ . \tag{3.7}$$

Proof: The situation is depicted in Figure 3.1. Substituting for x_1 using Equation (3.1a) we get

$$v_1 - b'_1x_1 = v_1 + b'_1v_o - b'_1(\rho_o x_o + m_o) \ . \tag{3.8}$$

Similarly,

$$v_2 - b'_2x_2 = v_2 + b'_2v_1 + b'_2\rho_1v_o - b'_2(\rho_1(\rho_o x_o + m_o) + m_1), \tag{3.9}$$

and at the i-th stage we have

$$\begin{aligned} v_i - b'_ix_i = {} & v_i + b'_iv_{i-1} + b'_i\rho_{i-1}v_{i-2} + \cdots + b'_i\,\rho_{i-1}\cdot\cdots\cdot\rho_1v_o \\ & - b'_i(\rho_{i-1}(\rho_{i-2}(\ldots.(\rho_o x_o + m_o)\ldots) + m_{i-2}) + m_{i-1}) \ . \end{aligned} \tag{3.10}$$

We now define

$$\begin{aligned} \tilde{x}_o &= x_o \\ \tilde{x}_{i+1} &= \rho_i\tilde{x}_i + m_i \quad , \quad i=0,1,\ldots, \end{aligned} \tag{3.11}$$

and

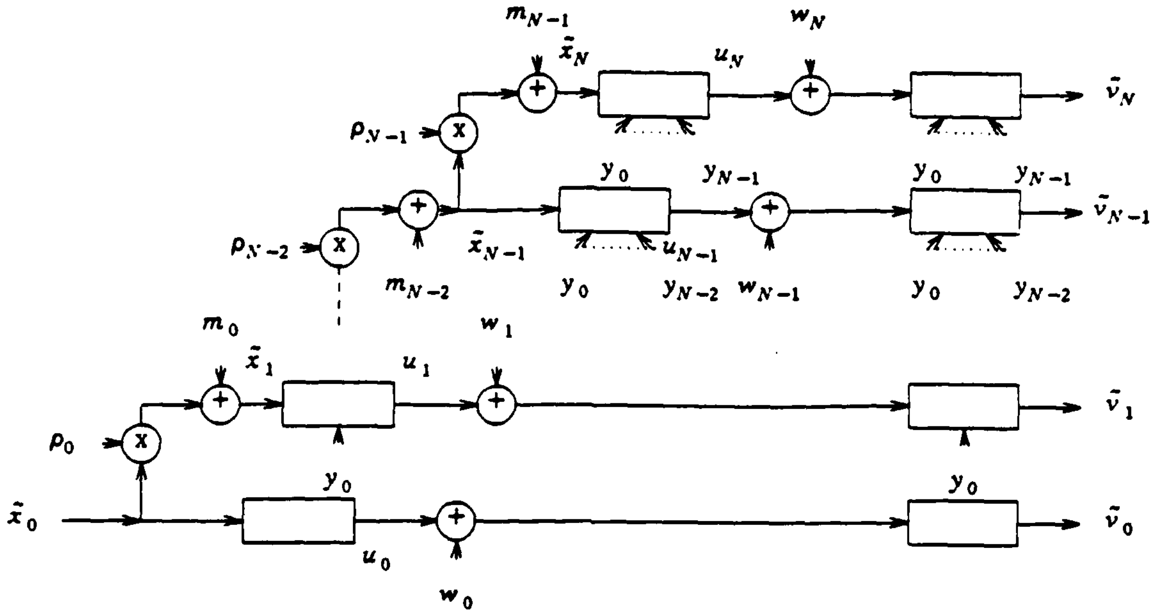

Figure 3.1. Diagrammatic representation of Problem P3.

$$\tilde{v}_i = v_i + b'_i v_{i-1} + b'_i \rho_{i-1} v_{i-2} + \ldots + b'_i \rho_{i-1} \cdot \cdots \cdot \rho_1 v_o \ . \tag{3.12}$$

Using these new variables, the cost function (3.3) can be rewritten as

$$\begin{gathered} J(h^N, \tilde{\gamma}^N) = E[\sum_{i=0}^{N} a'_i (\tilde{v}_i - b'_i \tilde{x}_i)^2] + c_N \ , \\ \tilde{v}_i = \tilde{\gamma}_i (y^i, v^{i-1}) \ , \end{gathered} \tag{3.13}$$

where the evolution of the $\tilde{x}_i$'s is determined by (3.11). Since this is a team problem, and for each fixed h^N the resulting stochastic control problem has classical information, minimization of (3.13) over $(h^N, \tilde{\gamma}^N)$ is equivalent to its minimization over (h^N, γ^N) where γ_i has only y^i as its argument.

The relationship between $\tilde{v}_i$'s and v_i's is given by

$$\tilde{v}^N = v^N B^T \tag{3.14}$$

where B^T is the transpose of the following nonsingular lower triangular matrix:

$$B = \begin{bmatrix} 1 & 0 & 0 & 0 & \cdot & 0 & 0 \\ b'_1 & 1 & 0 & 0 & \cdot & 0 & 0 \\ b'_2\rho_1 & b'_2 & 1 & 0 & \cdot & 0 & 0 \\ b'_3\rho_2\rho_1 & b'_3\rho_2 & b'_3 & 1 & \cdot & 0 & 0 \\ \vdots & & & & \cdot & 1 & 0 \\ b'_N\rho_{N-1}\cdot\cdot\rho_1 & b'_N\rho_{N-1}\cdot\cdot\rho_2 & \cdot & & \cdot\cdot & b'_N & 1 \end{bmatrix} .$$

We may thus write

$$v^N = \tilde{v}^N [B^T]^{-1} ,$$

and we thus see that in order to obtain the optimal solution to P1, we may equivalently solve the problem in terms of $\tilde{v}_i$ and $\tilde{x}_i$ (over h^N, γ^N), which is precisely Problem P3. This then leads to the following equivalence between the solutions of Problems P1 and P3:

Lemma 3.1.

(i) Problem P1 admits a solution *if*, and *only if*, Problem P3 does.

(ii) If (h^N, γ^N) is a solution for P1, then $(h^N, \gamma^N B^T)$ solves P3; conversely, if (h^N, γ^N) solves P3, then $(h^N, \gamma^N [B^T]^{-1})$ is a solution for P1.

□

3.4. An Auxiliary Problem

In this section we formulate and solve an auxiliary problem which will play an important role in the solution to Problem P3.

Consider the situation depicted in Figure 3.2, the problem being one of finding the signals $u_o, \ldots, u_N$ subject to the power constraints

$$E[u_i^2] \leqslant P_i^2 ,$$

so as to minimize the mean-square error in the estimation of z_N using $y_o, y_1, \ldots, y_N$. It is

given that z_o, n_i, w_i, i=0,1,..., are mutually independent Gaussian random variables, each with mean zero and variance indicated by $\sigma^2_{(\cdot)}$, the subscript being the identifier. Following the convention of Information Theory, we let $I(z_N;y^N)$ denote the mutual information between z_N and y^N, and call the supremum of this quantity the capacity of the corresponding system.

We shall first solve this problem for the case when N=1. The proof for arbitrary finite N will then be shown to follow a similar line of reasoning. The case with N=1 is illustrated in Figure 3.3.

Recall from the discussion in Section 1.3 that for any channel the minimum possible distortion D* that can result from its use is related to its capacity by

$$R(D^*) = C \ .$$

First consider the lower branch of Figure 3.3, redrawn as Figure 3.4. We find D* for this system. Let us suppose that the input is connected to a Gaussian memoryless source with variance $\sigma^2_{z_1}$, in which case we have

$$R(D) = \mathrm{Max}(0, \frac{1}{2}\log(\frac{\sigma^2_{z_1}}{D})) \ . \tag{3.15}$$

Let z_o^j denote the sequence $(z_{o1}, \ldots, z_{oj})$, which we again interpret as a row vector; define n_o^j, v_1^j and z_1^j likewise. We introduce

$$m_1^j \triangleq E(z_1^j \mid z_o^j) = \frac{(1+\alpha_1)\sigma^2_{z_1}}{(1+\alpha_1)^2\sigma^2_{z_1}+\sigma^2_{n_1}} \, z_o^j$$

and observe the following.

(a) The components of m_1^j are i.i.d. Gaussian variates with zero mean and variance

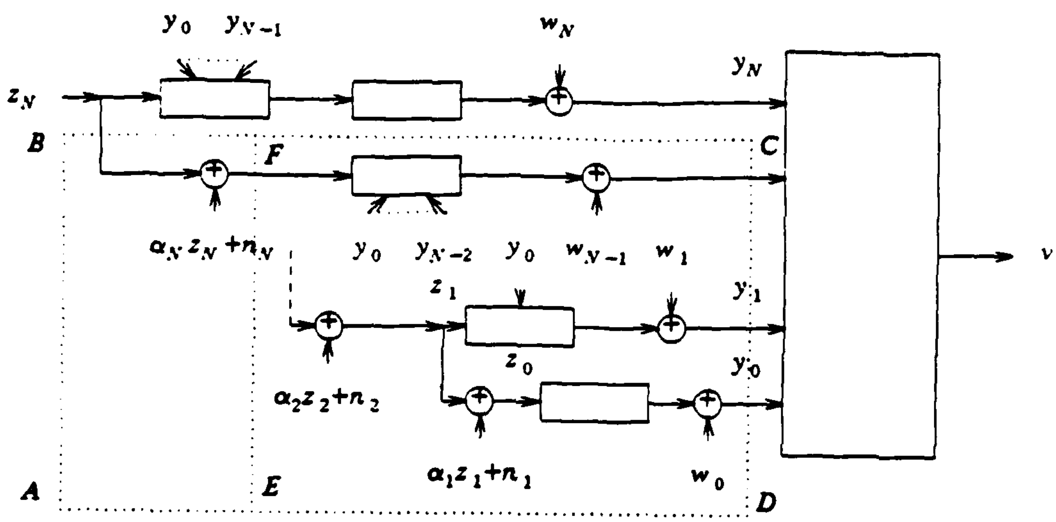

Figure 3.2. Diagrammatic representation of the auxiliary problem.

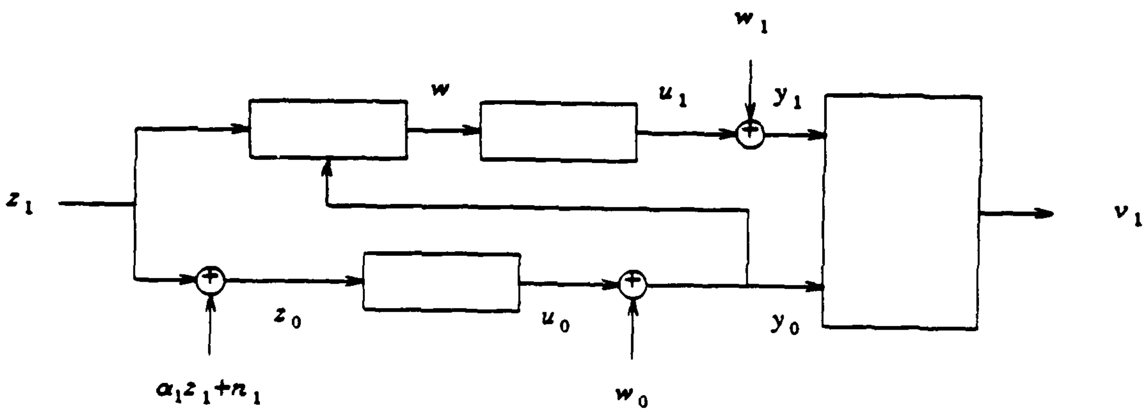

Figure 3.3. The auxiliary problem with N=1.

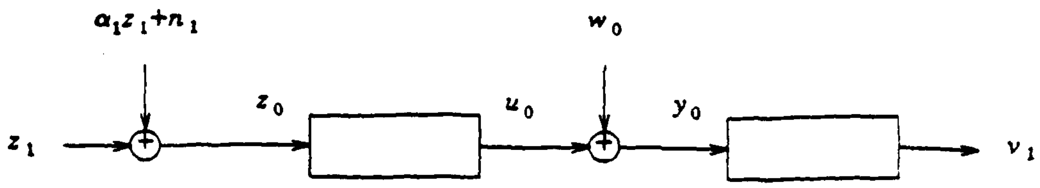

Figure 3.4. The lower branch of the system depicted in Figure 3.3.

$$\frac{[(1+\alpha_1)\sigma_{z_1}^2]^2}{(1+\alpha_1)^2\sigma_{z_1}^2+\sigma_{n_1}^2} \tag{3.16}$$

(b)

$$E\{\frac{1}{j}\|m_1^j-z_1^j\|^2\} = \frac{\sigma_{z_1}^2\sigma_{n_1}^2}{(1+\alpha_1)^2\sigma_{z_1}^2+\sigma_{n_1}^2}, \tag{3.17}$$

where $\|\cdot\|$ denotes the Euclidean norm. We also have

$$\lim_{j\to\infty} \mathrm{Inf} \; \frac{1}{j} \; E\|v_1^j-m_1^j\|^2 = \frac{(1+\alpha_1)^2\sigma_{z_1}^4}{(1+\alpha_1)^2\sigma_{z_1}^2+\sigma_{n_1}^2} \, e^{-2C_o} \tag{3.18}$$

where

$$C_o = \frac{1}{2}\log\left(\frac{P_o^2+\sigma_{w_o}^2}{\sigma_{w_o}^2}\right) \tag{3.19}$$

Now since

$$\frac{1}{j} E\|z_1^j-v_1^j\|^2 = \frac{1}{j} E\|z_1^j-m_1^j+m_1^j-v_1^j\|^2 = \frac{1}{j} E\|z_1^j-m_1^j\|^2 + \frac{1}{j}E\|m_1^j-v_1^j\|^2 .$$

we have

$$D^* = \lim_{j\to\infty} \frac{1}{j} E\|z_1^j - v_1^j\|^2$$

$$= \frac{\sigma_{z_1}^2 \sigma_{n_1}^2}{(1+\alpha_1)^2\sigma_{z_1}^2+\sigma_{n_1}^2} + \frac{(1+\alpha_1)^2\sigma_{z_1}^4}{((1+\alpha_1)^2\sigma_{z_1}^2+\sigma_{n_1}^2)} \frac{\sigma_{w_o}^2}{(P_o^2+\sigma_{w_o}^2)} \tag{3.20}$$

$R(D^*)$ may now be computed, and we have the following result:

Lemma 3.2: For the system depicted in Figure 3.4, the mutual information $I(z_1;y_o)$ is bounded above by

$$C_{eq} = \frac{1}{2} \log \left[\frac{(P_o^2+\sigma_{w_o}^2)((1+\alpha_1)^2\sigma_{z_1}^2+\sigma_{n_1}^2)}{\sigma_{n_1}^2(P_o^2+\sigma_{w_o}^2)+(1+\alpha_1)^2\sigma_{z_1}^2\sigma_{w_o}^2} \right] . \tag{3.21}$$

□

Now, consider the system depicted in Figure 3.3. We first note that

$$I(z_1;y_o,y_1) = I(z_1;y_o) + I(z_1;y_1 \mid y_o) . \tag{3.22}$$

Now,

$$I(z_1;y_1 \mid y_o) = I(z_1,w;y_1 \mid y_o) - I(w;y_1 \mid z_1,y_o) \tag{3.23a}$$

$$\leqslant I(z_1,w;y_1 \mid y_o) \tag{3.23b}$$

$$= I(w;y_1 \mid y_o) \tag{3.23c}$$

$$= H(y_1 \mid y_o) - H(y_1 \mid w,y_o) \tag{3.23d}$$

$$= H(y_1 \mid y_o) - H(y_1 \mid w) \tag{3.23e}$$

$$\leqslant H(y_1) - H(y_1 \mid w) \tag{3.23f}$$

$$= I(y_1;w) \tag{3.23g}$$

$$\leqslant \frac{1}{2} \log\left(\frac{P_1^2+\sigma_{w_1}^2}{\sigma_{w_1}^2}\right) \tag{3.23h}$$

where $H(\cdot)$ is the entropy and $H(\cdot \mid \cdot)$ the conditional entropy. Here steps (a), (d) and (g)

follow from the definition of mutual information, steps (c) and (e) are due to the Markov Property, step (b) follows because information is always positive, step (f) is valid because conditioning cannot increase entropy, and the last step holds because, for a fixed variance, the Gaussian random variable has the maximum entropy (Kagan et al. [1973]).

Using (3.21) and (3.23), along with (3.22), we obtain the next result.

Lemma 3.3: For the system depicted in Figure 3.3,

$$I(z_1;y_o,y_1) \leqslant \frac{1}{2}\log\left[\frac{(P_o^2+\sigma_{w_o}^2)((1+\alpha_1)^2\sigma_{z_1}^2+\sigma_{n_1}^2)}{(\sigma_{n_1}^2(P_o^2+\sigma_{w_o}^2)+(1+\alpha_1)^2\sigma_{z_1}^2\sigma_{w_o}^2)}\cdot\frac{P_1^2+\sigma_{w_1}^2}{\sigma_{w_1}^2}\right]. \tag{3.24}$$

□

Using this upper bound on $I(z_1,y_o,y_1)$ we can find a lower bound on the minimum mean-square error achievable when the problem is to estimate z_1 from the observations y_o and y_1.

We have

$$I(z_1;y_o,y_1) \geqslant I(z_1;v_1) \geqslant \frac{1}{2}\log\frac{\sigma_{z_1}^2}{E[(z_1-v_1)^2]} \tag{3.25}$$

which implies (using Equation (3.24))

$$\frac{1}{2}\log\frac{(P_o^2+\sigma_{w_o}^2)((1+\alpha_1)^2\sigma_{z_1}^2+\sigma_{n_1}^2)}{(\sigma_{n_1}^2(P_o^2+\sigma_{w_o}^2)+(1+\alpha_1)^2\sigma_{z_1}^2\sigma_{w_o}^2)}\cdot\frac{P_1^2+\sigma_{w_1}^2}{\sigma_{w_1}^2} \geqslant \frac{1}{2}\log\frac{\sigma_{z_1}^2}{E[(z_1-v_1)^2]} \tag{3.26}$$

i.e.,

$$E[(z_1-v_1)^2] \geqslant \frac{\sigma_{w_1}^2(\sigma_{n_1}^2(P_o^2+\sigma_{w_o}^2)+(1+\alpha_1)^2\sigma_{z_1}^2\sigma_{w_o}^2)}{(P_o^2+\sigma_{w_o}^2)(P_1^2+\sigma_{w_1}^2)\sigma_{z_o}^2}. \tag{3.27}$$

We next note that if we use the policy

$$\begin{aligned} u_o &= \lambda_o z_o \\ u_1 &= \lambda_1(z_1 - E(z_1|y_o)) \end{aligned} \tag{3.28}$$

with λ_o and λ_1 chosen so as to satisfy the power constraints, the minimum mean-square error is indeed achieved and we have

Lemma 3.4. The policies given by (3.28) are the policies which minimize the mean-square error in estimating z_1 from the pair (y_o, y_1).

We now consider the case with arbitrary N, depicted in Figure 3.2. First, consider the problem in the absence of the most recent observation, i.e., the uppermost channel of Figure 3.2 removed. Assuming that the version of the problem with (N-1) channels is already solved, the capacity C_{N-1} (i.e., maximum mutual information between input and output) for the portion of the system within the rectangular box CDEF is known. We can therefore find the minimum achievable distortion for a memoryless Gaussian source with variance $\sigma^2_{z_N}$ when only the portion within the box ABCD is in use, by computing the conditional estimate of z_N given z_{N-1}, and transmitting this optimally. Using this minimum achievable distortion we can find an upper bound for $I(z_N;y^{N-1})$ as follows (where z_N^j is the sequence $z_{N_1}, \ldots, z_{N_j}$)

$$\begin{aligned} D^* &= \lim_{j\to\infty} \frac{1}{j} E\|z_N^j - v_N^j\|^2 \\ &= \frac{\sigma^2_{z_N}\sigma^2_{n_N}}{(1+\alpha_N)^2\sigma^2_{z_N}+\sigma^2_{n_N}} + \frac{(1+\alpha_N)^2\sigma^4_{z_N}}{(1+\alpha_N)^2\sigma^2_{z_N}+\sigma^2_{n_N}} e^{-2C_{N-1}} \end{aligned} \tag{3.29}$$

i.e.,

$$I(z_N;y^{N-1}) \leqslant \frac{1}{2} \log \left| \frac{(1+\alpha_N)^2 \sigma_{z_N}^2 + \sigma_{n_N}^2}{\sigma_{n_N}^2 + (1+\alpha_N)^2 \sigma_{z_N}^2 e^{-2C_{N-1}}} \right| . \tag{3.30}$$

We can use a series of inequalities as in (3.23) to show that

$$I(z_N;y_N|y^{N-1}) \leqslant \frac{1}{2} \log \frac{P_N^2 + \sigma_{w_N}^2}{\sigma_{w_N}^2} . \tag{3.31}$$

Now since

$$I(z_N;y^N) = I(z_N;y^{N-1}) + I(z_N;y_N|y^{N-1}) \tag{3.32}$$

we get

$$I(z_N;y^N) \leqslant \frac{1}{2} \log \left| \frac{(1+\alpha_N)^2 \sigma_{z_N}^2 + \sigma_{n_N}^2}{\sigma_{n_N}^2 + (1+\alpha_N)^2 \sigma_{z_N}^2 e^{-2C_{N-1}}} \cdot \frac{P_N^2 + \sigma_{w_N}^2}{\sigma_{w_N}^2} \right| \tag{3.33}$$

$$\triangleq C_N$$

and we have the following lemma.

Lemma 3.5. The mutual information $I(z_N;y^N)$ is bounded above by C_N, which is the last step of the recursion:

$$C_o = \frac{1}{2} \log\left(\frac{P_o^2 + \sigma_{w_o}^2}{\sigma_{w_o}^2}\right) \tag{3.34a}$$

and for i=1,...,N

$$C_i = \frac{1}{2}\log\left[\frac{(1+\alpha_i)^2\sigma_{z_i}^2+\sigma_{n_i}^2}{\sigma_{n_i}^2+(1+\alpha_i)^2\sigma_{z_i}^2 e^{-2C_{i-1}}}\cdot\frac{P_i^2+\sigma_{w_i}^2}{\sigma_{w_i}^2}\right]. \tag{3.34b}$$

□

Let Δ_i denote the *minimum achievable mean-square error* when z_i is estimated using y^i, i.e., (using (3.25))

$$\Delta_i \triangleq \sigma_{z_i}^2 e^{-2C_i}. \tag{3.35}$$

We therefore have

$$\Delta_o = \frac{\sigma_{z_o}^2\sigma_{w_o}^2}{P_o^2+\sigma_{w_o}^2} \tag{3.36a}$$

and for i=1,...,N

$$\Delta_i = \frac{\sigma_{w_i}^2}{P_i^2+\sigma_{w_i}^2}\left[\frac{(1+\alpha_i)^2\sigma_{z_i}^4}{\sigma_{z_{i-1}}^4}\Delta_{i-1} + \frac{\sigma_{z_i}^2}{\sigma_{z_{i-1}}^2}\sigma_{n_i}^2\right]. \tag{3.36b}$$

We shall next show that this lower bound is tight and may be achieved by using the policies

$$u_o = \lambda_o z_o \tag{3.37a}$$

and for i=1,...,N

$$u_i = \lambda_i(z_i - E(z_i|y^{i-1})) . \tag{3.37b}$$

Here the λ_i's are chosen so as to meet the power constraints with equality. Since

$$z_{i-1} = (1+\alpha_i)z_i + n_i \tag{3.38}$$

we may equivalently write

$$z_i = \tilde{\rho}_{i-1} z_{i-1} + m_{i-1} \tag{3.39}$$

where $\tilde{\rho}_{i-1}$ is a constant,

$$\tilde{\rho}_{i-1} \triangleq (1+\alpha_i)\frac{\sigma_{z_i}^2}{\sigma_{z_{i-1}}^2} \ , \tag{3.40}$$

and m_{i-1} is a Gaussian random variable which is independent of z_{i-1} and has variance

$$\sigma_{m_{i-1}}^2 = \frac{\sigma_{z_i}^2}{\sigma_{z_{i-1}}^2}\,\sigma_{n_i}^2 . \tag{3.41}$$

With the policies chosen as in (3.37), we have

$$y_o = \lambda_o z_o + w_o \tag{3.42a}$$

and for i=1,...,N

$$y_i = \lambda_i(z_i - E(z_i|y^{i-1})) + w_i . \tag{3.42b}$$

Let Σ_i denote the mean-square error in the estimation of z_i from y^i when the communication strategies are chosen as in (3.37), i.e.,

$$\Sigma_i = E[(z_i - E(z_i|y^i))^2] \ .$$

We then have, using (3.36)

$$\lambda_o^2 = \frac{P_o^2}{\sigma_{z_o}^2} \tag{3.43a}$$

and for i=1,...,N

$$\lambda_i^2 = \frac{P_i^2}{(1+\alpha_i)^2 \frac{\sigma_{z_i}^4}{\sigma_{z_{i-1}}^4} \Sigma_{i-1} + \frac{\sigma_{z_i}^2}{\sigma_{z_{i-1}}^2} \sigma_{n_i}^2} . \tag{3.43b}$$

Further, by our specific choice of policy, $(z_i - E(z_i|y^{i-1}))$ is a zero mean Gaussian random variable, and therefore

$$E[(z_i - E(z_i|y^{i-1}) - E[(z_i - E(z_i|y^{i-1}))|y_i])^2] = \frac{s_i^2 \sigma_{w_i}^2}{P_i^2 + \sigma_{w_i}^2} \tag{3.44}$$

where

$$\begin{aligned} s_i^2 &\triangleq E[(z_i - E(z_i|y^{i-1}))^2] \\ &= (1+\alpha_i)^2 \frac{\sigma_{z_i}^4}{\sigma_{z_{i-1}}^4} \Sigma_{i-1} + \frac{\sigma_{z_i}^2}{\sigma_{z_{i-1}}^2} \sigma_{n_i}^2 . \end{aligned} \tag{3.45}$$

Also, $E(E(z_i|y^{i-1})|y_i)=0$ by our choice of policy, since y_i is independent of y^{i-1}, and therefore the expression on the left-hand side of (3.44) becomes

$$\begin{aligned} &E[(z_i - E(z_i|y^{i-1}) - E(z_i|y_i))^2] \\ &= E[(z_i - E(z_i|y^i))^2] \\ &= \Sigma_i \end{aligned}$$

and we get the recursion (for i=1,...,N)

$$\Sigma_i = \frac{\sigma_{w_i}^2}{P_i^2 + \sigma_{w_i}^2} \left[\frac{(1+\alpha_i)^2 \sigma_{z_i}^4}{\sigma_{z_{i-1}}^4} \Sigma_{i-1} + \frac{\sigma_{z_i}^2}{\sigma_{z_{i-1}}^2} \sigma_{n_i}^2 \right] \tag{3.46a}$$

with the initial condition

$$\Sigma_o = \frac{\sigma^2_{w_o}\sigma^2_{z_o}}{P_o^2+\sigma^2_{w_o}}. \tag{3.46b}$$

The recursion for the Σ_i's is therefore identical to the recursion for the Δ's (given by (3.36)), which denote the minimum mean-square error achievable. This shows that the lower bound on error is indeed tight and thus leads to the following theorem.

Theorem 3.1.

(a) The policies given by (3.37) minimize the mean-square error incurred in estimating z_N from y^N for the system depicted in Figure 3.2, where the λ_i's are defined by (3.43) using the Σ_i's defined by (3.46).

(b) The *minimum* mean-square error is given by the last step of the recursion (3.46) or equivalently by the last step of the recursion (3.36).

□

3.5. Solutions to Problems P3 and P1

We now return to Problem P3 defined in Section 3.3, where the policies

$$u_i = h_i(\tilde{x}_i, y^{i-1})$$

and

$$v_i = \gamma_i(y^i)$$

are to be chosen in order to minimize

$$J = E[\sum_{i=0}^{N} a'_i(\tilde{v}_i - b'_i\tilde{x}_i)^2] \tag{3.47}$$

under the constraints depicted in Figure 3.1.

We first consider the minimization of the N-th term in the expression for J, which is

$$E[a'_N(\tilde{v}_N - b'_N \tilde{x}_N)^2] ,$$

the optimization problem being equivalent to minimizing

$$E[a'_N b'^2_N (\tilde{v}'_N - \tilde{x}_N)^2] ,$$

where

$$\tilde{v}'_N \equiv \frac{\tilde{v}_N}{b'_N} ,$$

i.e., the problem is one of forming the best estimate of x_N under the mean-square distortion criterion. We now show that the situations depicted in Figures 3.1 and 3.2 are identical except for nomenclature. To show this equivalence we note that for i=1,...,N

$$z_{i-1} = (1+\alpha_i) z_i + n_i \tag{3.48}$$

which implies

$$z_i = (1+\alpha_i) \frac{\sigma^2_{z_i}}{\sigma^2_{z_{i-1}}} z_{i-1} + m_{i-1} \tag{3.49}$$

where the m_i's are zero mean Gaussian random variables each with variance

$$\sigma^2_{m_i} = \frac{\sigma^2_{z_{i+1}}}{\sigma^2_{z_i}} \sigma^2_{n_{i+1}} . \tag{3.50}$$

We therefore have

$$\rho_{i-1} = (1+\alpha_i) \frac{\sigma^2_{z_i}}{\sigma^2_{z_{i-1}}} \text{ for } i=1,...,N$$

and by defining $\sigma^2_{\tilde{x}_o} = \sigma^2_{z_o}$ we can complete the correspondence between the variables $\tilde{x}_i$'s and z_i's for $i=0,1,\ldots,N$.

The solution to the problem of minimizing the mean-square error in estimating $\tilde{x}_N$ from $y_o, \ldots, y_N$ may therefore be obtained as in Section 3.4. Using the techniques of Section 3.4 we get

(a) The minimum mean-square error in estimating x_i using y^i is given by Δ_i, where Δ_i's satisfy the recursion (for $i=1,\ldots,N$)

$$\Delta_i = \frac{\sigma^2_{w_i}}{P_i^2+\sigma^2_{w_i}} (\rho^2_{i-1}\Delta_{i-1}+\sigma^2_{m_{i-1}}) \tag{3.51a}$$

with the initial condition

$$\Delta_o = \frac{\sigma^2_{\tilde{x}_o}\sigma^2_{w_o}}{P_o^2+\sigma^2_{w_o}} . \tag{3.51b}$$

(b) The optimal encoding strategies are

$$u_o = h^*_o(\tilde{x}_o) = \lambda_o\tilde{x}_o \tag{3.52a}$$

and for $i=1,\ldots,N$

$$u_i = h^*_i(\tilde{x}^i,y^{i-1}) = \lambda_i(\tilde{x}_i - E(\tilde{x}_i|y^{i-1})) \tag{3.52b}$$

where the λ_i's satisfy the recursion (for $i=1,\ldots,N$)

$$\lambda_i^2 = \frac{P_i^2}{\rho^2_{i-1}\Delta_{i-1}+\sigma^2_{m_{i-1}}} \tag{3.53a}$$

with the initial condition

$$\lambda_o^2 = \frac{P_o^2}{\sigma_{\tilde{x}_o}^2} \tag{3.53b}$$

(the Δ_i's being as defined by (3.51)).

(c) The optimal choice for the $\tilde{\gamma}_i$'s is

$$\tilde{v}_i = b'_i E(\tilde{x}_i | y^i) \tag{3.54}$$

where $E(\tilde{x}_i|y^i)$ satisfy the recursion (for i=1,...,N)

$$E(\tilde{x}_i|y^i) = \rho_{i-1}E(\tilde{x}_{i-1}|y^{i-1}) + \frac{P_i}{P_i^2+\sigma_{w_i}^2}(\rho_{i-1}^2\Delta_{i-1}+\sigma_{m_{i-1}}^2)^{1/2}y_i \tag{3.55a}$$

with the initial condition

$$E(\tilde{x}_o|y^o) = \frac{P_o\sigma_{\tilde{x}_o}y_o}{P_o^2+\sigma_{\tilde{x}_o}^2} \tag{3.55b}$$

(the Δ_i's being as defined in (3.51)).

We finally note that the policies which minimize the mean-square error in the estimation of $\tilde{x}_i$ given y^i for i=0, N-1, are identical to the corresponding policies used in the estimation of $\tilde{x}_N$ given y^N, and we therefore have the following theorem.

Theorem 3.2.

(a) The optimum policies h_i and $\tilde{\gamma}_i$ for Problem P3 are given by (3.52) and (3.54), respectively, using the λ_i's and Δ_i's as defined by (3.53) and (3.51), respectively.

(b) The minimum value of the cost function for Problem P3 is

$$J^* = \sum_{i=0}^{N} a'_i b'^2_i \Delta_i + c_N .$$

□

Now we turn to the original Problem P1 formulated in Section 3.2. Taking the difference

$$\frac{\tilde{v}_i}{b'_i} - \rho_{i-1} \frac{\tilde{v}_{i-1}}{b'_{i-1}} \tag{3.56}$$

and using (3.54), (3.55) and (3.12), we find that

$$\frac{v_i}{b'_i} = (\rho_{i-1} - b'_{i-1}) \frac{v_{i-1}}{b'_{i-1}} + \frac{P_i}{P_i^2 + \sigma^2_{w_i}} (\rho^2_{i-1}\Delta_{i-1} + \sigma^2_{m_{i-1}})^{1/2} y_i \tag{3.57}$$

which implies that the optimal control policies for the original problem are

$$v^*_i = \gamma^*_i(y^i) = b'_i\, E(x_i|y^i) \triangleq b'_i \hat{x}_i \tag{3.58}$$

where $\hat{x}_i \triangleq E(x_i|y^i)$ satisfy the recursion (for $i=1,\dots,N$)

$$\hat{x}_i = (\rho_{i-1} - b'_{i-1})\hat{x}_{i-1} + \frac{P_i}{P_i^2 + \sigma^2_{w_i}} (\rho^2_{i-1}\Delta_{i-1} + \sigma^2_{m_{i-1}})^{1/2} y_i \tag{3.59a}$$

with the initial condition

$$\hat{x}_o = \frac{P_o \sigma_{x_o}}{P_o^2 + \sigma^2_{x_o}} y_o , \tag{3.59b}$$

the Δ_i's being as defined in (3.51).

We therefore have the following theorem.

Theorem 3.3.

(a) The optimum policies $\{h^*_i\}$ and $\{\gamma^*_i\}$ for Problem P1 are given by (3.52) and (3.58), respectively, using the λ_i's and Δ_i's as defined by (3.53) and (3.51), respectively.

(b) The minimum value of the cost function for Problem P1 is

$$J^* = \sum_{i=0}^{N} a'_i b'^2_i \Delta_i + c_N .$$

□

An Illustration

Consider the case with N=2, the stochastic system model being given as

$$\begin{aligned} x_1 &= x_0 + m_0 - v_0 \\ x_2 &= x_1 + m_1 - v_1 \\ x_3 &= x_2 + m_2 - v_2 , \end{aligned}$$

and the objective being to minimize

$$J(h^2,\gamma^2) = \sum_{i=0}^{2} (x_{i+1}^2 + v_i^2) ,$$

subject to the constraints of Equations (3.1b) through (3.1e), it being given that

$$\sigma_{x_0}^2 = \sigma_{w_i}^2 = \sigma_{m_i}^2 = P_i^2 = 1.0 .$$

Using Claim 3.1, (Section 3.2), we get the equivalent cost functional

$$\sum_{i=0}^{2} a'_i (v_i - b'_i x_i)^2 + c_2$$

where

$$a'_0 = 13/5\ ,\ a'_1 = 5/2\ ,\ a'_2 = 2\ ,$$
$$b'_0 = 8/13\ ,\ b'_1 = 3/5\ ,\ b'_2 = 1/2\ ,$$

and

$$c_2 = 613/130\ .$$

Using Theorem 3.3 above, we obtain the optimal strategies for the problem:

$$u^*_0 = \lambda_0 x_0$$
$$u^*_1 = \lambda_1(x_1 - E(x_1 \mid y^0))$$
$$u^*_2 = \lambda_2(x_2 - E(x_2 \mid y^1))$$

where

$$\lambda_0^2 = 1\ ,\ \lambda_1^2 = 2/3 \text{ and } \lambda_2^2 = 4/7\ ,$$

and

$$v^*_0 = 8/3\ \hat{x}_0$$
$$v^*_1 = 3/5\ \hat{x}_1$$
$$v^*_2 = 1/2\ \hat{x}_2$$

where

$$\hat{x}_0 = 1/2\ y_0$$
$$\hat{x}_1 = 5/26\ y_0 + 3/34\ y_1$$
$$\hat{x}_2 = 1/3\ y_0 + 3/2\ y_1 + 7/16\ y_2\ .$$
$$E(x_1 \mid y^0) = 7/26\ y_0$$

and

$$E(x_2 \mid y^1) = 1/13\ y_0 + 3/10\ y_1\ .$$

Further, the optimal cost is 6.1852.

□

3.6. The Soft Constraint Version

We consider again the problem depicted in Figure 3.1, but now with the power constraints

$$E[u_i^2] \leqslant P_i^2$$

removed, i.e., the optimum power levels are also to be determined via the underlying optimization problem. Such a formulation is useful in situations where more "costly" measurements that contain more useful or reliable information may be used. It may be possible to transmit a larger power at the encoder (at additional cost) in order to further decrease the mean-square error at the decoder, this tradeoff being reflected by the cost criterion. Mathematically, we may represent this as a power constraint which is "implied" or "soft", appearing as an additional term in the cost functional, which now becomes

$$J(h^N,\gamma^N) = E[\sum_{i=0}^{N} (a'_i(v_i - b'_i x_i)^2 + q_i u_i^2)] . \tag{3.60}$$

We shall obtain the solution to the soft constraint version by using the solution to the hard constraint Problem P1 found in Section 3.5. Let J_P denote the infimum of J under the hard power constraints,

$$J_P \triangleq \operatorname*{Inf}_{h^N;\gamma^N;E[h_i^2]=P_i^2, i \in P} J(h^N,\gamma^N) \tag{3.61}$$

We then have the following series of equalities and inequalities:

$$
\begin{aligned}
J_P &= \sum_{i=0}^{N} q_i P_i^2 + \inf_{E[h_i^2]=P_i^2} E\left[\sum_{i=0}^{N} (a'_i(v_i - b'_i x_i)^2)\right] \\
&\geqslant \sum_{i=0}^{N} q_i P_i^2 + \inf_{E[h_i^2]\leqslant P_i^2} E\left[\sum_{i=0}^{N} (a'_i(v_i - b'_i x_i)^2)\right] \\
&= \sum_{i=0}^{N} q_i P_i^2 + \sum_{i=0}^{N} a'_i b'^2_i \Delta_i \qquad (3.62) \\
&\geqslant \min_{P_i^2 \geqslant 0} \left[\sum_{i=0}^{N} q_i P_i^2 + a'_i b'^2_i \Delta_i\right] \\
&= \sum_{i=0}^{N} q_i P*_i^2 + a' b'^2_i \Delta*_i
\end{aligned}
$$

where the Δ_i's are defined in (3.51) and $\Delta*$'s are defined recursively likewise, with P_i replaced by $P*_i$, i.e., for $i=1,...,N$

$$
\Delta*_i = \frac{\sigma^2_{w_i}}{P*_i^2 + \sigma^2_{w_i}} (\rho^2_{i-1} \Delta*_{i-1} + \sigma^2_{m_{i-1}}) \qquad (3.63a)
$$

with the initial condition

$$
\Delta*_o = \frac{\sigma^2_{w_o} \sigma^2_{x_o}}{P*_o^2 + \sigma^2_{w_o}} . \qquad (3.63b)
$$

In order to find the optimal power levels ($P*_i^2$'s), we can solve the following problem

$$
\min_{P_o, \ldots, P_N} \sum_{i=0}^{N} q_i P_i^2 + a'_i b'^2_i \Delta_i \qquad (3.64)
$$

which is a nonlinear optimal control problem, the solution to which is given by the following dynamic program (with $\rho_{-1} \triangleq 1$ and $\sigma^2_{m_{-1}} \triangleq 0$), where $W_i(\Delta)$ is the "optimum cost to go" given that the system is at state Δ at stage i

$$W_{N+1} = 0$$

$$W_i(\Delta) = \underset{P_i^2}{\mathrm{Min}} \left[q_i P_i^2 + a'_i b'^2_i \frac{\sigma^2_{w_i}}{P_i^2 + \sigma^2_{w_i}} (\rho^2_{i-1}\Delta + \sigma^2_{m_{i-1}}) + W_{i+1}\left(\frac{\sigma^2_{w_i}}{P_i^2 + \sigma^2_{w_i}} (\rho^2_{i-1}\Delta + \sigma^2_{m_{i-1}}) \right) \right] \tag{3.65}$$

and the optimal value of the cost is

$$\underset{P_o^2, \ldots, P_N^2}{\mathrm{Min}} \; J(P_o^2, \ldots, P_N^2) = W_o(\sigma^2_{x_o}) . \tag{3.66}$$

We next show that a solution to the above problem always exists. If we define

$$f(\Delta, P_i^2) \triangleq q_i P_i^2 + \frac{a'_i b'^2_i \sigma^2_{w_i}}{P_i^2 + \sigma^2_{w_i}} (\rho^2_{i-1}\Delta + \sigma^2_{m_{i-1}}) + W_{i+1} \left(\frac{\sigma^2_{w_i}(\rho^2_{i-1}\Delta + \sigma^2_{m_{i-1}})}{P_i^2 + \sigma^2_{w_i}} \right) \tag{3.67}$$

then

$$W_i(\Delta) = \underset{P_i^2}{\mathrm{Min}} \; f(\Delta, P_i^2) .$$

Note that W_i is a continuous function of its argument if W_{i+1} is, since the continuity of W_{i+1} implies continuity of f. From the continuity of W_{N+1}, (which was defined to be zero), the continuity of W_i follows for all i. We also note that as $P_i^2 \to \infty$, $f(\Delta, P_i^2) \to \infty$ also, and since $P_i^2 \geqslant 0$, the search for P^{*2}_i can be confined to a compact set over which a continuous function always admits a minimum.

The dynamic program (3.65) can therefore be solved, yielding values for

$$P^*_o, P^*_1, \ldots, P^*_N ,$$

and we have the following theorem.

Theorem 3.4. Consider the problem

$$\underset{h^N,\gamma^N}{\text{Minimize}}\ J(h^N,\gamma^N)\ ,$$

subject to (3.1a) through (3.1e) where $J(h^N,\gamma^N)$ is defined by Equation (3.60).

(a) The optimum policies $\{h^*_i\}$ and $\{\gamma^*_i\}$ for this problem are given by (3.52) and (3.58), respectively, using λ_i's and Δ_i's as defined by Equations (3.51) and (3.53), with the solution to the dynamic program (3.65) providing the optimum power levels, i.e.,

$$P_i^2 = P^{*2}_i \text{ for } i=0,\ldots,N\ .$$

(b) The optimum cost is given by

$$W_o(\sigma_{x_o}^2) = \sum_{i=0}^{N} q_i P^{*2}_i + a'_i b'^2_i \Delta^*_i\ ,$$

with the Δ^*_i's being defined by (3.63).

□

An Illustration

The optimal power levels depend critically on the power penalties (q_i's). If for the problem stated at the beginning of this section we assume N=1 and the following parameter values

$$\sigma_{x_o}^2 = 1.0\ ,\ \sigma_{w_o}^2 = 1.0\ ,\ \sigma_{w_1}^2 = 1.0\ ,\ \sigma_{m_o}^2 = 1.0$$
$$q_o = 2.0\ ,\ q_1 = 4.0\ ,\ \rho_o = 0.5\ ,\ a'_o = 1.0$$
$$b'_o = 1.0\ ,\ a'_1 = 2.0 \text{ and } b'_1 = 1.0\ ,$$

then the optimal value of the cost is 3.5, which is attained by $P_o^2=P_1^2=0.0$. If the power penalty q_o is changed to 0.25 with all other parameters remaining the same, we can achieve an optimal cost of 2.9747, which is attained by using $P_o^2 = 1.4495$ and $P_1^2 = 0.0$.

If the power penalty q_1 is also changed to 0.25, then the optimal cost is further reduced to 1.9968, which is attained by $P_0^2 = 1.1609$ and $P_1^2 = 1.9876$. It is notable that the optimal solution satisfies a threshold property, and the number of channels in use depends on the relative magnitudes of the weighting terms.

3.7. The Infinite Horizon Problem

Notation

Let x_i denote the realization of a first-order Markov process:

$$x_{i+1} = \rho x_i + m_i \,. \tag{3.68}$$

Here x_0, m_0, m_1,... are zero mean Gaussian random variables:

$$x_0 \sim N(0,\sigma_x^2)$$
$$m_i \sim N(0,\sigma_m^2) \,.$$

The measurement y_i is a noise corrupted version of the control u_i:

$$y_i = u_i + w_i \tag{3.69}$$

where w_0, w_1,.. are zero mean Gaussian random variables:

$$w_i \sim N(0,\sigma_w^2)$$

We are concerned with obtaining the optimal solution to Problem P^∞ below.

Problem $\mathbf{P}^\infty$

$$\underset{h^{\infty},\gamma^\infty}{\text{Minimize}}\ J(h^\infty,\gamma^\infty) = E[\sum_{i=0}^{\infty}(qu_i^2+a(v_i-x_i)^2)\beta^i]$$

where a, q are given positive constants, β is the given discount factor $(0<\beta<1)$ and

$$u_i = h_i(x_i, y^{i-1}) \tag{3.70a}$$

$$v_i = \gamma_i(y^i) \,. \tag{3.70b}$$

□

We treat the infinite horizon problem as a limit of the finite horizon case with horizon length N, as $N \to \infty$. This may be done provided that the discounted cost remains bounded and the optimum policy sequence converges to a well-defined limit.

The truncated version of the finite horizon problem, with horizon length N, is given as Problem P^N below:

Problem P^N

$$\underset{h^N,\gamma^N}{\text{Minimize}}\ J_N(h^N,\gamma^N) = E[\sum_{i=0}^{N}(qu_i^{\,2}+a(v_i-x_i)^2)\beta^i]$$

subject to (3.68) through (3.70).

□

For notational convenience, let $q'_i \triangleq q\beta^i$ and $a'_i \triangleq a\beta^i$.

We first consider the following hard constraint version of Problem P^N.

Problem PH^N

$$\underset{h^N,\gamma^N}{\text{Minimize}}\ E[\sum_{i=0}^{N} a'_i(v_i-x_i)^2]$$

subject to

$$E[u_i^{\,2}] \leqslant P_i^{\,2}$$

under the constraint (3.68) through (3.70).

□

The following Lemma now follows directly from the analysis in the preceding sections.

Lemma 3.6: (a) Problem PH^N admits an optimal solution which is linear in the measurements and is given as follows:

$$u^*_i = h^*_i(x_i, y^{i-1}) = \lambda_i(x_i - E(x_i \mid y^{i-1}))$$
$$v^*_i = \gamma^*_i(y^i) = \hat{x}_i$$

where

$$\lambda_0^2 = \frac{P_0^2}{\sigma_w^2}, \tag{3.71a}$$

$$\lambda_i^2 = \frac{P_i^2}{\rho^2 \Delta_{i-1} + \sigma_m^2} \text{ for } i=1,\ldots,N, \tag{3.71b}$$

the Δ_i's satisfy

$$\Delta_0 = \frac{\sigma_x^2 \sigma_w^2}{P_o^2 + \sigma_w^2}, \tag{3.72a}$$

$$\Delta_i = \frac{\sigma_w^2}{P_i^2 + \sigma_w^2} (\rho^2 \Delta_{i-1} + \sigma_m^2) \text{ for } i=1,\ldots,N. \tag{3.72b}$$

and $\hat{x}_i = E(x_i \mid y^i)$ satisfies the recursion

$$\hat{x}_0 = \frac{P_0\sigma_x}{P_0^2+\sigma_x^2} y_0 ,$$

$$\hat{x}_i = \rho\hat{x}_{i-1} + \frac{P_i}{P_i^2+\sigma_w^2} (\rho^2\Delta_i+\sigma_m^2)^{1/2} y_i .$$

(b) The minimum value of the cost is

$$J^* = \sum_{i=0}^{N} a'_i\Delta_i .$$

□

Now let J_P denote the infimum of $J_N(h^N,\gamma^N)$ under the hard power constraints, i.e.,

$$J_P \triangleq \inf_{h^N,\gamma^N,E[h_i^2]=P_i^2} J_N(h^N,\gamma^N) .$$

Using a sequence of equalities and inequalities as in (3.62), we have

$$J_p = \sum_{i=0}^{N}(qP_i^{2*}+a\Delta^*_i)\beta^i$$

where the Δ_i's are as defined in (3.72), and Δ^*_i's are defined likewise, with P_i^2 replaced by P_i^{2*}.

The next task is to find the optimum power levels $\{P_i^{2*}\text{'s}\}$, which is done via the following deterministic optimal control problem:

The Deterministic N-stage Problem

$$\underset{P_0^2,\ldots,P_N^2}{\text{Minimize}} \sum_{i=0}^{N} (qP_i^2 + a\Delta_i)\beta^i \tag{3.73}$$

subject to (3.72a) and (3.72b).

Notation: For each positive scalar Δ, let $\{W_k(\Delta)\}_{k=0}^{N+1}$ be defined recursively by

$$W_{N+1}(\Delta) = 0 \tag{3.74a}$$

$$W_k(\Delta) = \inf_{P^2}[qP^2 + \frac{a\sigma_w^2}{P^2+\sigma_w^2}(\rho^2\Delta+\sigma_m^2) + \beta W_{k+1}\left(\frac{\sigma_w^2}{P^2+\sigma_w^2}(\rho^2\Delta+\sigma_m^2)\right)] \text{ for } k=N,\ldots,0\,. \tag{3.74b}$$

Let $P_k^2 = P_k^2(\Delta)$ be a minimizing solution of the right-hand side of (3.74b), whenever it exists, and let $\{\Delta^*_k\}_{k=0}^N$ be the trajectory sequence defined recursively by

$$\Delta^*_0 = \left(\frac{\sigma_w^2\sigma_x^2}{P_0^2(\sigma_x^2)+\sigma_w^2}\right) \tag{3.75a}$$

$$\Delta^*_k = \frac{\sigma_w^2}{P_k^2(\Delta^*_{k-1})+\sigma_w^2}\,(\rho^2\Delta^*_{k-1} + \sigma_m^2)\,. \tag{3.75b}$$

Finally, let

$$P_0^{2*} = P_0^2(\sigma_x^2) \tag{3.76a}$$

$$P_k^{2*} = P_k^2(\Delta^*_{k-1})\,. \tag{3.76b}$$

Proposition 3.1:

(i) The minimization problem (3.74b) admits a solution for each positive Δ.

(ii) The control problem (3.73) admits a solution $\{P_k^{2*}\}_{k=0}^N$ which is given by (3.76) and the corresponding optimal trajectory is generated by (3.75).

(iii) The minimum value for the optimal control problem is $W_0(\sigma_x^2)$.

The proof of the proposition above follows from the following lemmata.

Lemma 3.7: The value of the optimal control problem (3.73) is $J^* = W_0(\sigma_x^2)$ where $W_0(\cdot)$ is obtained through the recursive equations

$$W_{N+1}(\Delta) = 0 \tag{3.77a}$$

$$W_k(\Delta) = \inf_{P^2}[qP^2 + \frac{a\sigma_w^2}{P^2+\sigma_w^2}(\rho^2\Delta+\sigma_m^2)+ \beta W_{k+1}(\frac{\sigma_w^2}{P^2+\sigma_w^2}(\rho^2\Delta + \sigma_m^2))] \text{ for } k \leqslant N\,. \quad (3.77b)$$

Furthermore, if the right-hand side of (3.77b) admits a solution sequence $\{P_k^2\}$, $k \leqslant N$, then

$$P_0^2(\sigma_x^2)$$

and

$$P_k^2(\Delta^*_{k-1}) \text{ for } 1 \leqslant k \leqslant N$$

provide the optimal solution, where Δ^*_k is generated by (3.75).

Proof: This follows from a standard dynamic programming argument.

□

Lemma 3.8: For every $\Delta > 0$, there exists a solution to the right-hand side of (3.74b).

Proof: Define $f(\Delta,P^2)$ by

$$f_k(\Delta,P^2) = qP^2 + \frac{a\sigma_w^2}{P^2+\sigma_w^2} + W_{k+1}(\frac{\sigma_w^2}{P^2+\sigma_w^2}(\rho^2\Delta + \sigma_m^2)) \quad (3.78)$$

so that

$$W_k(\Delta) = \inf_{P^2} f_k(\Delta,P^2)\,.$$

Note that W_k, which is positive for all k, is a continuous function of its argument if W_{k+1} is, since the continuity of W_{k+1} implies continuity of f_k. From the continuity of W_{N+1} (which is defined to be zero) the continuity of W_k follows for all k. We then note that as $P^2 \to \infty$, so does $f(\Delta,P^2)$, and since $P^2 \geqslant 0$, the search for an optimum may be confined to a compact set over which a continuous function always admits a minimum.

□

The following lemmata will be used in the construction of the solution for the infinite horizon problem.

Lemma 3.9: $W_k(\Delta)$ is strictly increasing for decreasing k, for all $\Delta>0$, i.e., $W_k(\Delta) > W_{k+1}(\Delta)\ \forall k \leqslant N$.

Proof: Clearly the lemma is true for k=N, since $W_{N+1} \triangleq 0$ and $W_N(\Delta)$ is necessarily larger than zero for all Δ. We now note the following sequence of equalities and inequalities:

$$
\begin{aligned}
& W_k(\Delta) - W_{k+1}(\Delta) \\
& = \underset{P^2>0}{\mathrm{Min}}[qP^2 + \frac{a\sigma_w^2}{P^2+\sigma_w^2}(\rho^2\Delta + \sigma_m^2) + \beta W_{k+1}(\frac{\sigma_2^2}{P^2+\sigma_w^2}(\rho^2\Delta + \sigma_m^2))] \\
& - \underset{P^2>0}{\mathrm{Min}}[qP^2 + \frac{a\sigma_w^2}{P^2+\sigma_w^2}(\rho^2\Delta + \sigma_m^2) + \beta W_{k+2}(\frac{\sigma_w^2}{P^2+\sigma_w^2}(\rho^2\Delta + \sigma_m^2))] \\
& \geqslant q\hat{P}^2 + \frac{a\sigma_w^2}{\hat{P}^2+\sigma_w^2}(\rho^2\Delta + \sigma_m^2) + \beta W_{k+1}(\frac{\sigma_w^2}{\hat{P}^2+\sigma_w^2}(\rho^2\Delta + \sigma_m^2)) \\
& - q\hat{P}^2 + \frac{a\sigma_w^2}{\hat{P}^2+\sigma_w^2}(\rho^2\Delta + \sigma_m^2) - \beta W_{k+2}(\frac{\sigma_w^2}{\hat{P}^2+\sigma_w^2}(\rho^2\Delta + \sigma_m^2)) \\
& = \beta(W_{k+1} - W_{k+2}) \cdot (\frac{\sigma_w^2(\rho^2\Delta+\sigma_m^2)}{\hat{P}^2+\sigma_w^2})
\end{aligned}
$$

where $\hat{P}^2$ is chosen as the argument of the first minimization. (In case of nonunique solutions, any one of the minimizing solutions may be chosen).

Thus if $W_{k+1}(\Delta)$ is larger than $W_{k+2}(\Delta)$, then $W_k(\Delta)$ is larger than $W_{k+1}(\Delta)$. Since $W_N(\Delta)$ is known to be larger than $W_{N+1}(\Delta)$, the proof is complete.

□

Lemma 3.10: $W_k(\Delta)$ is an increasing function of Δ $\{W_k(\Delta) \uparrow \Delta\}$ for all $k \leqslant N$.

Proof: We prove this by induction. First consider the case with k=N. We have

$$W_N(\Delta) = \underset{P^2 \geq 0}{\mathrm{Min}}[qP^2 + \frac{a\sigma_w^2}{P^2+\sigma_w^2}(\rho^2\Delta + \sigma_m^2)] .$$

There are only two possible cases, either $P^{2*} = 0$, or $P^{2*} > 0$. If $P^{2*} = 0$, then $W_N(\Delta) = a(\rho^2\Delta + \sigma_m^2)$. If $P^{2*} > 0$, which requires

$$q - \frac{a\sigma_w^2(\rho^2\Delta+\sigma_m^2)}{(P^{2*}+\sigma_w^2)^2} = 0$$

i.e.,

$$P^{2*} = [\frac{a\sigma_w^2(\rho^2\Delta+\sigma_m^2)}{q}]^{1/2} - \sigma_w^2 \tag{3.79}$$

and we get

$$W_N(\Delta) = 2a^{1/2}q^{1/2}\sigma_w(\rho^2\Delta + \sigma_m^2)^{1/2} - q\sigma_w^2 . \tag{3.80}$$

Thus the lemma is true for k=N.

Now, if $W_{k+1}(\Delta) \uparrow \Delta$, then for each P^2

$$W_{k+1}(\frac{a\sigma_w^2}{P^2+\sigma_w^2}(\rho^2\Delta + \sigma_m^2)) \uparrow \Delta ,$$

since

$$\frac{a\sigma_w^2}{P^2+\sigma_w^2}(\rho^2\Delta + \sigma_m^2) \uparrow \Delta .$$

Also,

$$(qP^2 + \frac{a\sigma_w^2(\rho^2\Delta+\sigma_m^2)}{P^2+\sigma_w^2}) \uparrow \Delta ,$$

and thus both terms in the expression to be minimized to obtain $W_K(\Delta)$ are increasing in Δ

for all P^2. Therefore, $W_k(\Delta) \uparrow \Delta$.

□

Lemma 3.11: For each $\Delta > 0$, $W_k(\Delta)$ is bounded above for all k, by an affine function of Δ, i.e.,

$$0 < W_k(\Delta) \leqslant \Omega_{1,k}\Delta + \Omega_{2,k} \,.$$

Proof: The proof is by induction, using the observation that since $W_k(\Delta)$ is given by the minimum over P^2, an upper bound is given by the value that the expression to be minimized attains when P^2 is fixed arbitrarily at zero.

Thus $W_N(\Delta) \leqslant a(\rho^2\Delta+\sigma_m^2)$ and we may choose $\Omega_{1,N} = a\rho^2, \Omega_{2,N} = a\sigma_m^2$.

Now consider the following sequence of equalities and inequalities

$$\begin{aligned}
W_k(\Delta) &= \operatorname*{Min}_{P^2}[qP^2 + \frac{a\sigma_w^2}{P^2+\sigma_w^2}(\rho^2\Delta + \sigma_m^2) + \beta W_{k+1}(\frac{\sigma_w^2}{P^2+\sigma_w^2}(\rho^2\Delta + \sigma_m^2))] \\
&\leqslant \operatorname*{Min}_{P^2}[qP^2 + \frac{a\sigma_w^2}{P^2+\sigma_w^2}(\rho^2\Delta + \sigma_m^2) + \beta W_{k+1}(\rho^2\Delta + \sigma_m^2)] \\
&\leqslant a(\rho^2\Delta + \sigma_m^2) + \beta W_{k+1}(\rho^2\Delta + \sigma_m^2) \\
&\leqslant a(\rho^2\Delta + \sigma_m^2) + \beta(\Omega_{1,k+1}(\rho^2\Delta + \sigma_m^2) + \Omega_{2,k+1}) \\
&= (a\rho^2 + \beta\Omega_{1,k+1}\rho^2)\Delta + a\sigma_m^2 + \beta\Omega_{1,k+1}\sigma_m^2 + \beta\Omega_{2,k+1} \\
&\triangleq \Omega_{1,k}\Delta + \Omega_{2,k} \,.
\end{aligned}$$

Therefore, the lemma is proved with the sequences $\{\Omega_{1,k}\}$ and $\{\Omega_{2,k}\}$ defined recursively by

$$\Omega_{1,N} = a\rho^2 \quad , \quad \Omega_{2,N} = a\sigma_m^2$$
$$\Omega_{1,k} = a\rho^2 + \beta\rho^2\Omega_{1,k+1}$$
$$\Omega_{2,k} = a\sigma_m^2 + \beta\sigma_m^2\Omega_{1,k+1} + \beta\Omega_{2,k+1} .$$

□

We now return to the study of the infinite horizon problem. Note that since the optimal policy for the stochastic control problem is linear, the stationary limiting policy is given by

$$h^*_n(x_n, y^{n-1}) = \lambda^*(x_n - E(x_n \mid y^{n-1})) \tag{3.81}$$

where

$$\lambda^{*2} = \frac{P^{2*}}{\rho^2\Delta^* + \sigma_m^2} , \tag{3.82}$$

with P^{2*} and Δ^* being obtained through the stationary solution of the optimum control problem as $N \to \infty$.

For each N, denote the solution given by Proposition 3.1. (ii) by $\{P_k^{2*}\}^N$, $k<N$. We then expect that

$$P^{2*} = \lim_{N\to\infty} \{P_k^{2*}\}^N$$

for every finite k.

To establish the existence of this limit, we recall that $W_k(\Delta)$ is strictly increasing for decreasing $k<N$ (Lemma 3.9) and further that it is bounded above by an affine function (Lemma 3.11). This last property follows since both Ω_{1k} and Ω_{2k} are bounded in retrograde time,

$$\Omega_{1k} < \left(\frac{a\rho^2}{1-\beta\rho^2}\right) \text{ and } \Omega_{2,k} < \frac{a\sigma_m^2}{(1-\beta)(1-\rho^2\beta)} . \tag{3.83}$$

Hence, $\lim\limits_{k\to\infty} W_k(\Delta) = W(\Delta)$ where the limiting function satisfies

$$W(\Delta) = \min_{P^2} \left[qP^2 + \frac{a\sigma_w^2(\rho^2\Delta+\sigma_m^2)}{P^2+\sigma_w^2} + \beta W\left[\frac{\sigma_w^2}{P^2+\sigma_w^2}(\rho^2\Delta+\sigma_m^2)\right]\right] . \tag{3.84}$$

Denote the minimizing solution here by $P^2(\Delta)$. Now note that since $\rho^2<1$, Equation (3.75b) describes a stable system with P_k^2 replaced by P^2, and hence $\Delta_k \to \Delta^*$ where Δ^* solves

$$\Delta^* = \frac{\sigma_w^2}{P^2(\Delta^*)+\sigma_w^2}(\rho^2\Delta^*+\sigma_m^2) . \tag{3.85}$$

Let

$$P^{2*} \equiv P^2(\Delta^*) . \tag{3.86}$$

Then we have the following solution to the infinite horizon problem:

Theorem 3.5: With $N\to\infty$ in (3.73), the stochastic control problem in consideration admits the optimal stationary policies,

$$h^*_n = \lambda^*(x_n - E(x_n \mid y^{n-1}))$$

for n sufficiently large, where

$$\lambda^{*2} = \frac{P^{2*}}{\rho^2\Delta^*+\sigma_m^2} ,$$

with Δ^* and P^{2*} given by (3.85) and (3.86). □

To numerically compute the optimal stationary policies, we start with $\Delta_0 = 0$ and run the following algorithm:

Algorithm A:

(1) Compute

$$(P_k^2)^* = \arg \min_{P^2 \geqslant 0} [qP^2 + \frac{a\sigma_w^2(\rho^2\Delta_k + \sigma_m^2)}{P^2 + \sigma_w^2} + \frac{\beta(qP^2 + a\Delta_k)}{(1-\beta)}]$$

(2) Compute the new value, Δ_{k+1}, by

$$\Delta_{k+1} = \frac{\sigma_w^2}{(P_k^2)^* + \sigma_w^2} (\rho^2\Delta_k + \sigma_m^2)$$

(3) Go to step (1), and iterate.

□

The optimal cost is then given by

$$W(\Delta^*) = (q(P^2)^* + \frac{a\sigma_w^2(\rho^2\Delta^* + \sigma_m^2)}{(P^2)^* + \sigma_w^2})/(1-\beta).$$

□

We next show that Algorithm A always converges. We first note that $(P_k^2)^*$, found from step (1) of the algorithm, satisfies

$$(P_k^2)^* = \mathrm{Max}\{0, (\frac{a^{1/2}(1-\beta)^{1/2}}{q^{1/2}} \sigma_w(\rho^2\Delta_k + \sigma_m^2)^{1/2} - \sigma_w^2)\}$$

which implies that if $\Delta_{k+1} \geqslant \Delta_k$, then $(P_{k+1}^2)^* \geqslant (P_k^2)^*$.

Now, given $\Delta_k > \Delta_{k-1}$, we have

$$
\begin{aligned}
&\Delta_{k+1} - \Delta_k \\
&= \frac{\sigma_w^2}{(P_k^2)^* + \sigma_w^2}(\rho^2\Delta_k + \sigma_m^2) - \frac{\sigma_w^2}{(P_{k-1}^2)^* + \sigma_w^2}(\rho^2\Delta_{k-1} + \sigma_m^2) \\
&\geqslant \frac{\sigma_w^2}{(P_k^2)^* + \sigma_w^2}(\rho^2\Delta_k + \sigma_m^2) - \frac{\sigma_w^2}{(P_k^2)^* + \sigma_w^2}(\rho^2\Delta_{k-1} + \sigma_m^2) \\
&= \frac{\sigma_w^2}{(P_k^2)^* + \sigma_w^2}\,\rho^2(\Delta_k - \Delta_{k-1}) \\
&> 0 .
\end{aligned}
$$

Therefore, since $\Delta_1 > \Delta_0$, it follows that the Δ_k's form a monotone increasing sequence. Further, to show that the Δ_k's are bounded above, we consider the sequence

$$
\begin{aligned}
\Gamma_0 &= 0 \\
\Gamma_{k+1} &= \rho^2\Gamma_k + \sigma_m^2 ,
\end{aligned}
$$

and note that if $\Gamma_k \geqslant \Delta_k$, we have

$$
\begin{aligned}
\Gamma_{k+1} &= \rho^2\Gamma_k + \sigma_m^2 \\
&\geqslant \rho^2\Delta_k + \sigma_m^2 \\
&\geqslant \frac{\sigma_w^2}{(P_k^2)^* + \sigma_w^2}(\rho^2\Delta_k + \sigma_m^2) \\
&= \Delta_{k+1} ,
\end{aligned}
$$

i.e., if $\Gamma_k \geqslant \Delta_k$, then $\Gamma_{k+1} \geqslant \Delta_{k+1}$. But $\Gamma_0 = \Delta_0 (\equiv 0)$, and the sequence Γ_k is bounded above by

$$
\frac{\sigma_m^2}{(1-\rho^2)} .
$$

Therefore, the monotone sequence Δ_k is also bounded above, and the convergence of the algorithm follows.

Tables 3.1 through 3.4 provide a few of these convergence results for given parameter values.

TABLE 3.1. CONVERGENCE RESULTS FOR THE PROBLEM WITH $a = \sigma_w^2 = \sigma_m^2 = 1.0$, $\rho^2 = \beta = q = 0.5$.

Iter. #	Δ	$(P^2)^*$
0	0.0	0.0
1	1.0	0.22474
2	1.22474	0.26979
3	1.26979	0.27863
4	1.27863	0.28035
5	1.28035	0.26069
6	1.28069	0.28076
7	1.28076	0.28077
8	1.28077	0.28077
9	1.28077	0.28077

TABLE 3.2. CONVERGENCE RESULTS FOR THE PROBLEM WITH $a = \sigma_m^2 = 1.0$, $\rho^2 = \beta = 0.5$, $q = \sigma_w^2 = 0.1$.

Iter. #	Δ	$(P^2)^*$
0	0.0	0.60710
1	0.14142	0.63167
2	0.14633	0.63251
3	0.14650	0.63254
4	0.14650	0.63254

TABLE 3.3. CONVERGENCE RESULTS FOR THE PROBLEM WITH $a = \sigma_m^2 = 1.0$, $\rho^2 = \beta = 0.5$, $q = 0.1$, $\sigma_w^2 = 2.0$.

Iter. #	Δ	$(P^\lambda)^*$
0	0.0	1.16227
1	0.63245	1.62798
2	0.72559	1.69161
3	0.73832	1.70021
4	0.74004	1.70138
5	0.74027	1.70153
6	0.74030	1.70155
7	0.74031	1.70156
8	0.74031	1.70156

TABLE 3.4. CONVERGENCE RESULTS FOR THE PROBLEM WITH $\rho^2 = \sigma_m^2 = \beta = 0.5$, $a = \sigma_w^2 = 1.0$, $q = 0.1$.

Iter. #	Δ	$(P^\lambda)^*$
0	0.0	0.58113
1	0.31622	0.81399
2	0.36279	0.84580
3	0.36916	0.85010
4	0.37002	0.85069
5	0.37013	0.85076
6	0.37015	0.85078
7	0.37015	0.85078

We finally note that an infinite horizon version of the originally formulated Problem P1, can be solved via an equivalent problem of the form P^{∞}.

Consider the problem of minimizing $J(h^{\infty},\gamma^{\infty})$ where

$$J(h^{\infty},\gamma^{\infty}) = E[\sum_{i=0}^{\infty} \beta^i(qu_i^2 + a\beta x_{i+1}^2 + bv_i^2)]$$

subject to (3.1b), (3.1d) and (3.1e), where

$$x_{i+1} = \rho x_i + m_i - v_i .$$

Using (3.4) and (3.5), we find that the cost can be rewritten as

$$J = \sum_{i=0}^{\infty} q\beta^i u_i^2 + \sum_{i=0}^{\infty} a'_i(\tilde{v}_i - b'_i\tilde{x}_i)^2 ,$$

where

$$a'_i \rightarrow \beta^i(b+k\beta) \equiv a'\beta^i$$

and

$$b'_i \rightarrow \frac{k\beta\rho}{a'} \equiv b' ,$$

with the k found by solving for the positive root of the equation:

$$(k-a)(k\beta+b) = kb\beta\rho^2 ,$$

i.e., the k satisfies

$$k = \frac{1}{2\beta}(\sqrt{(b-b\beta\rho^2-a\beta)^2+4ab\beta} - (b-b\beta\rho^2-a\beta)) .$$

Thus, an infinite horizon version of the originally formulated problem, with discounted cost, may be solved by solving a problem of the form P^{∞}, with a replaced by $a'(b')^2$.

3.8. Conclusion

In this chapter we have studied the problem of simultaneous communication and control for first-order ARMA models with feedback. These are stochastic team problems, where the design of the measurement strategy itself is a part of the problem. As such, they are difficult to analyze because of the nonclassical nature of the information structure. For cases with hard power constraints on the measurement strategies, we have shown that the optimum measurement policy is to linearly amplify the innovation at each stage, to the maximum permissible power level. For the cases with soft power constraints the structure of the solution is similar; however, now the optimal design of the power levels is also a part of the problem. These optimum power levels may be found via dynamic programming. We have then studied some infinite horizon stochastic team problems involving a first-order ARMA model, established the existence of optimal stationary policies for these problems, provided an algorithm that always converges to the optimal solution, and also provided some numerical examples illustrating the calculation of these stationary policies.

CHAPTER 4

SIMULTANEOUS COMMUNICATION AND CONTROL: GENERAL ARMA MODELS WITH FEEDBACK

4.1. Introduction

In this chapter we study the problem of simultaneously designing communication and control strategies for problems involving ARMA models of orders higher than one. In Section 4.2 we formulate the general problem and then the transformed problem consisting of squared differences. In Section 4.3 we study one of the simplest such problems involving a second-order ARMA model, and show that the strategies which are optimal over the linear class may be outperformed by appropriately chosen nonlinear strategies. In Section 4.4 we consider optimality over the affine class, and show that within this class the optimal policy consists of transmitting only the current innovation, multiplied by a gain factor. In Section 4.5 we study the linear solutions for the second-order ARMA model; this illustrates the methodology and concepts for the more complex problem involving the general model which is treated in Section 4.6. The concluding remarks in Section 4.7 then end this chapter.

4.2. Problem Formulation

In Chapter 3 we had studied a stochastic dynamic system involving a first-order ARMA model, with the current state directly correlated only with the immediately preceding state. In case we allow this correlation to extend to j previous stages, we obtain a jth-order ARMA model. Accordingly, let us suppose that the stochastic system is specified by the following set of equations:

$$x_{i+1} = \sum_{k=0}^{j-1} \rho_{i+1,i-k} x_{i-k} + m_i - v_i \tag{4.1}$$

and

$$y_i = u_i + w_i \tag{4.2}$$

along with

$$u_i = h_i(x_i, y^{i-1}) \tag{4.3}$$

$$v_i = \gamma_i(y^i)\,. \tag{4.4}$$

Here (4.1) and (4.2) are the state and measurement equations respectively, the random variables x_0, w_i and m_i $(i \geqslant 0)$ are assumed to be independent, zero-mean and Gaussian with variances $\sigma^2_{(\cdot)}$, (the subscript being the identifier), and $\rho_{k,j} \equiv 0$ for $j<0$. The functions h_i and γ_i, $i \geqslant 0$, are the communication and control policies, respectively, each Borel measurable in its arguments, and leading to second-order random variables u_i and v_i, respectively.

As seen in Chapter 3, we may formulate a *hard* constraint version of the problem, by restricting the communication policies to satisfy

$$E[u_i^2] \leqslant P_i^2 \tag{4.5}$$

or the *soft* constraint version may be formulated, using the additional term

$$E[\sum_i q_i u_i^2] \tag{4.6}$$

in the cost functional, implying that a tradeoff between higher signalling costs and lower estimation costs is permissible.

These soft and hard constraint versions are given below as Problems PS° and PH°, respectively.

Problem PS°

$$\underset{h^N,\gamma^N}{\text{Minimize}}\ E[\sum_{i=0}^{N} q_i u_i^{2} + c_{i+1} x_{i+1}^{2} + d_i v_i^{2}]$$

subject to (4.1) through (4.4).

Problem PH°

$$\underset{h^N,\gamma^N}{\text{Minimize}}\ E[\sum_{i=0}^{N} c_{i+1} x_{i+1}^{2} + d_i v_i^{2}]$$

subject to (4.1) through (4.5).

Using completion of squares and a redefinition of the v_i's, (as in the case of the first-order ARMA model) we may obtain th^ equivalent problems PS and PH below:

Problem PS

$$\underset{h^N,\gamma^N}{\text{Minimize}}\ E[\sum_{i=0}^{N} q_i u_i^{2} + a_i(\tilde{v}_i - \sum_{k=0}^{j-1} b_{i,i-k}\tilde{x}_{i-k})^2]$$

subject to (4.2) through (4.4), with x_i replaced by $\tilde{x}_i$ and v_i replaced by $\tilde{v}_i$.

Problem PH

$$\underset{h^N,\gamma^N}{\text{Minimize}}\ E[a_i(\tilde{v}_i - \sum_{k=0}^{j-1} b_{i,i-k}\tilde{x}_{i-k})^2]$$

subject to (4.2) through (4.5), with x_i replaced by $\tilde{x}_i$ and v_i replaced by $\tilde{v}_i$. For these transformed problems, the $\tilde{x}_i$'s are given by the following recursion:

$$\tilde{x}_{i+1} = \sum_{k=0}^{j-1} \rho_{i+1,i-k}\tilde{x}_{i-k} + m_i \tag{4.7}$$

where $\tilde{x}_0 = x_0$ is given, $\rho_{k,j} = 0$ for all $j<0$ and $b_{k,j} = 0$ for $j<0$. Furthermore, the precise (recursive) expressions for the a_i's are given in Appendix A, where the details of the justification for this reformulation can also be found.

We now turn to analyzing these reformulated stochastic team problems.

4.3. Nonoptimality of Linear Laws

In this section we show that for one of the simplest team problems of the type above, involving an ARMA model of order 2, the optimum linear solution may be outperformed by an appropriately chosen nonlinear policy.

We first restrict our attention to the following stochastic team Problem P_a, of which a schematic representation is provided in Figure 4.1.

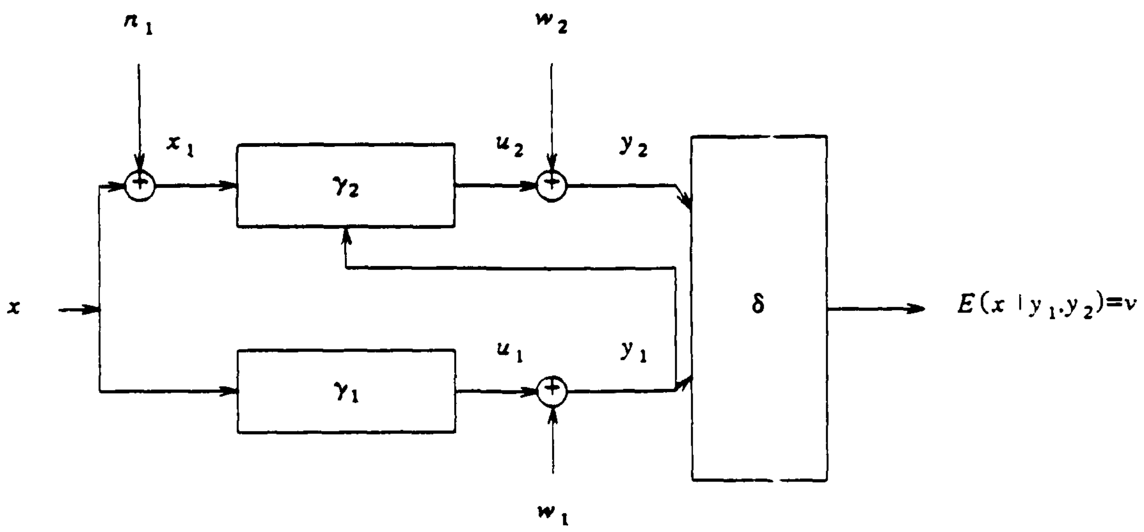

Figure 4.1. Schematics for Problem P_a.

Problem P_a

$$\underset{\gamma_1,\gamma_2,\delta}{\text{Minimize}}\ E[(x-v)^2]$$

where

$$x_1 = x + n_1 \tag{4.8}$$

$$u_1 = \gamma_1(x) \tag{4.9}$$

$$y_1 = u_1 + w_1 \tag{4.10}$$

$$u_2 = \gamma_2(x_1,y_1) \tag{4.11}$$

$$y_2 = u_2 + w_2 \tag{4.12}$$

and

$$v = \delta(y_1,y_2) \tag{4.13}$$

subject to the hard power constraints:

$$E[u_1^2] \leqslant P_1^2 \tag{4.14a}$$

$$E[u_2^2] \leqslant P_2^2\,. \tag{4.14b}$$

Note that if the problem involved estimating x_1 at the decoder (instead of x), then we would have had the two-stage version of a problem involving a first-order ARMA model as studied in Chapter 3, for which the optimal solutions have been shown to be linear.

We now show that Problem P_a above does not, in general, admit an optimal linear solution. This is done by constructing an instance of the problem where the optimal linear strategies are outperformed by appropriately chosen nonlinear strategies.

In order to see why one might suspect nonoptimality of affine laws, consider the above problem with $\sigma_{w_2}^2 = 0$. We then have Problem P'_a below which is represented

schematically in Figure 4.2, and for which the hard power constraint on u_2 is immaterial since there is no noise to combat.

Problem P'_a

$$\text{Minimize } E[(x-v)^2]$$

subject to (4.8) through (4.11), and (4.13), along with the restriction

$$y_2 = u_2 \,. \tag{4.15}$$

□

Note that since

$$v = \delta(y_1, y_2) \tag{4.16}$$

where

$$y_2 = u_2 = \gamma_2(x_1, y_1)$$

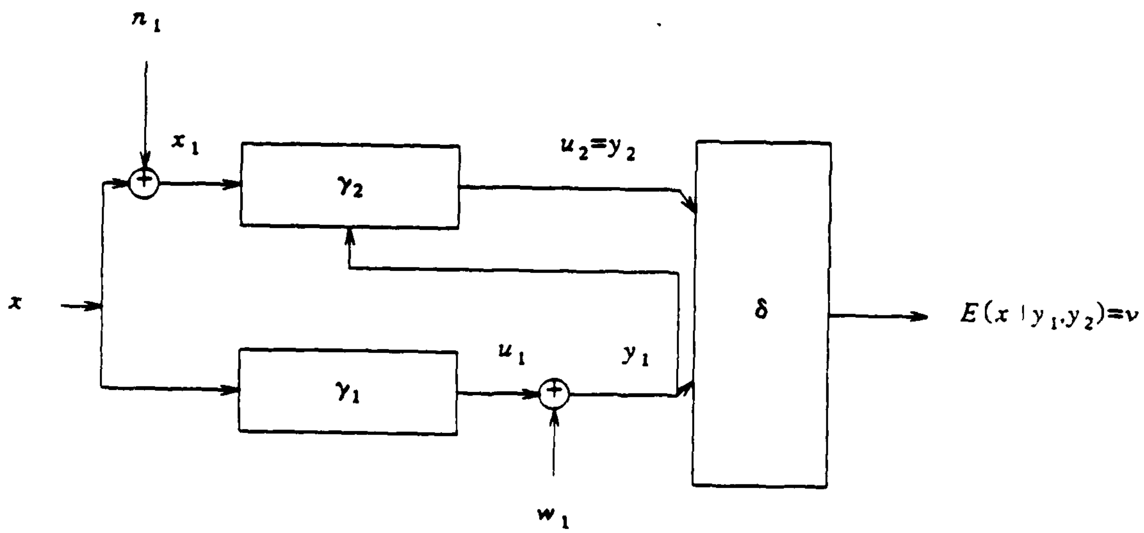

Figure 4.2. Schematics for Problem P'_a.

we may equivalently write

$$v = \delta(y_1, \gamma_2(x_1, y_1)) \tag{4.17a}$$

$$= \delta'(y_1, x_1)\,, \tag{4.17b}$$

and thus Problem P'_a may equivalently be represented as in Figure 4.3 below.

We thus obtain a problem of simultaneously designing encoding and decoding policies *with side information at the decoder*, for which nonlinear strategies that outperform the optimal linear strategies do exist (see Appendix B).

Since linear policies are not optimal for Problem P_a with $\sigma^2_{w_2} = 0$, they may continue to be nonoptimal for small enough values of $\sigma^2_{w_2}$. We show next that this is precisely the case. In particular, if we consider the optimal linear design for Problem P_a, using

$$u_1 = \gamma_1(x) = \lambda_1 x$$
$$u_2 = \gamma_2(x_1, y_1) = \lambda_2(x_1 - E(x_1 \mid y_1))$$

where λ_1 and λ_2 are chosen to meet the hard power constraints with equality, (this being the optimal choice in the affine class, as to be shown later), we have

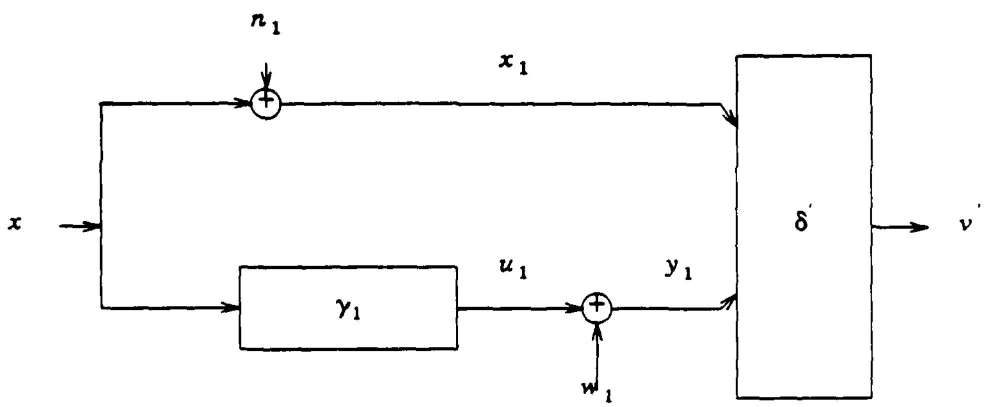

Figure 4.3. Equivalent representation for Problem P'_a.

$$E(x_1 \mid y_1) = \frac{\lambda_1 \sigma_x^2}{P_1^2 + \sigma_{w_1}^2}\, y_1 , \tag{4.18}$$

and

$$x_1 - E(x_1 \mid y_1) = n_1 + \frac{\sigma_{w_1}^2}{P_1^2 + \sigma_{w_1}^2}\, x - \frac{\lambda_1 \sigma_x^2}{P_1^2 + \sigma_{w_1}^2}\, w_1 \tag{4.19}$$

which implies that

$$u_2 = \lambda_2 \left[\frac{\sigma_{w_1}^2}{P_1^2 + \sigma_{w_1}^2}\, x - \frac{\lambda_1 \sigma_x^2}{P_1^2 + \sigma_{w_1}^2}\, w + n_1 \right]$$

with

$$\lambda_2^2 = \frac{P_2^2 (P_1^2 + \sigma_{w_1}^2)}{\sigma_x^2 \sigma_{w_1}^2 + \sigma_{n_1}^2 (P_1^2 + \sigma_{w_1}^2)} .$$

The mean square error in estimating x from the simultaneous observation of y_1 and y_2 then is

$$E[(x - E(x \mid y_1, y_2))^2] = \frac{\sigma_x^2 \sigma_{w_1}^2}{(P_1^2 + \sigma_{w_1}^2)(P_2^2 + \sigma_{w_2}^2)} \left[\sigma_{w_2}^2 + \frac{P_2^2 \sigma_{n_1}^2}{(\sigma_{n_1}^2 + \sigma_x^2 \sigma_{w_1}^2 / (P_1^2 + \sigma_{w_1}^2))} \right] . \tag{4.20}$$

Considering the situation with $\sigma_x^2 = 100.0$, $\sigma_{n_1}^2 = 0.99$, $\sigma_{w_1}^2 = 1.0$, $\sigma_{w_2}^2 = 0.01$, $P_1^2 = 85.0423$ and $P_2^2 = 100.99$, we find that the optimal linear policy yields a cost of 0.53467.

We next consider the design

$$\gamma_1(x) = x + \epsilon \operatorname{sgn} x$$
$$\gamma_2(x_1) = x_1$$

and

$$\delta(y_1,y_2) = \begin{cases} (y_1+y_2-\epsilon)/2 & \text{if } \; y_2 \geqslant 0 \\ y_1+y_2+\epsilon)/2 & \text{if } \; y_2 < 0 \end{cases}$$

(letting $\epsilon = -1.0$ we obtain $E[u_1^2] = 85.0423$ and $E(u_2^2) = 100.99$).

Now

$$y_2 = x + n_1 + w_2$$
$$\equiv x + w_3$$

where

$$w_3 \sim N(0,1)\,.$$

If we calculate the mean square error under the above policy by an analysis similar to that used in Appendix B, we find that the nonlinear policy yields a cost of 0.53172, and hence is superior to the optimal linear policy.

We now return to the problem of showing nonoptimality of linear laws for at least some instances of higher-order ARMA models. Consider the following second-order model with feedback, illustrated in Figure 4.4. We have

$$x_2 = \rho_{21}x_1 + \rho_{20}x_0 + m_1$$
$$x_1 = \rho_{10}x_0 + m_0$$

and x_0, m_0 and m_1 are given independent, zero-mean, Gaussian random variables.

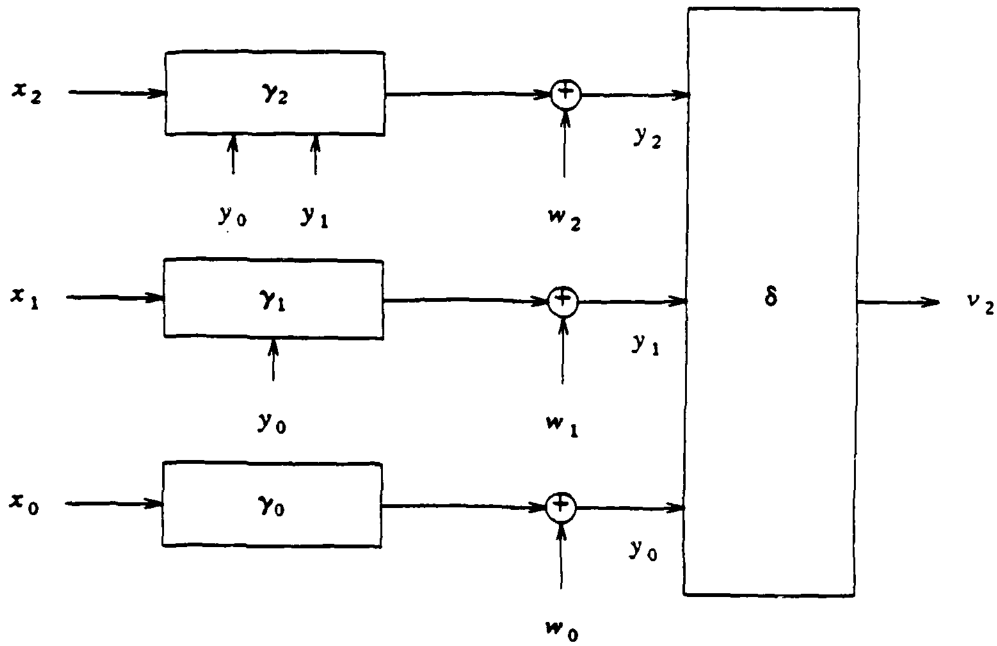

Figure 4.4. Schematics for a second-order model with feedback.

The problem is to minimize $E[(v_2-x_2)^2]$ under the schematics of Figure 4.4, with

$$v_2 = \gamma_2(y_0, y_1, y_2) .$$

Let us now suppose that $\sigma^2_{w_2}$ is arbitrarily large, essentially making the third channel redundant, and therefore

$$E(x_2 \mid y_0, y_1, y_2) = E(x_2 \mid y_0, y_1) .$$

Further suppose that

$$\rho_{21} = 0 \, , \, \rho_{20} = 1 \, , \, \sigma^2_{m_2} = 0 \, , \, \rho_{10} = 1 \, ,$$

which imply

$$\begin{aligned} x_2 &= x_0 \\ x_1 &= x_0 + m_1 \end{aligned}$$

and we obtain the problem depicted in Figure 4.5. We thus obtain a problem of the type

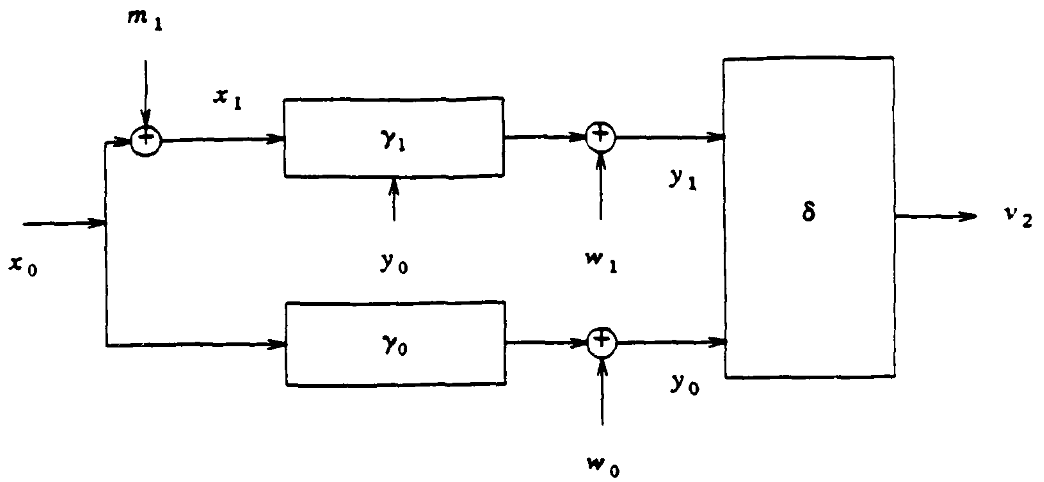

Figure 4.5. The second-order ARMA model under the given restrictions.

P_a, discussed earlier in this section, for which there are instances when linear strategies are not optimal. Hence we see that there are instances of the general problem (described by second-order ARMA processes) for which optimal linear designs are not optimal in the general class of policies.

4.4. Optimality over the Affine Class

In the preceding section we have shown that for ARMA models of order higher than one, the strategies which simultaneously provide optimum communication and control are not necessarily linear. In this section we show that if we confine the design to the affine class, then the optimal communication strategies use a linear transformation on the innovation process.

Theorem 4.1. Consider the general formulation of the Problem PS given in Section 4.2, but with h_i resticted to the class

$$u_i = h_i(\tilde{x}_i, y^{i-1}) = L_i(\tilde{x}_i, y^{i-1})$$

where L_i is a general affine mapping. Then one may, without loss of generality, be

confined to optimizing over the class of transmitter policies which satisfy the structural restriction

$$u_i = \lambda_i(\tilde{x}_i - E(\tilde{x}_i \mid y^{i-1}))\,. \tag{4.21}$$

Thus it is sufficient to optimize over the class of policies which use a *linear transformation on the innovation in* $\tilde{x}_i$.

Proof. Note that over the affine class we may write

$$u_i = L_i(\tilde{x}_i, y^{i-1}) = \tilde{u}_i + p_i$$

with

$$\tilde{u}_i = \lambda_i(\tilde{x}_i - E(\tilde{x}_i \mid y^{i-1}))$$

and

$$p_i = L'_i(y^{i-1})$$

where L'_i is an arbitrary affine mapping, and $\tilde{u}_i$ and p_i are uncorrelated. Thus, we have

$$\begin{aligned}
\operatorname*{Min}_{\gamma_N} J(h^N,\gamma^N) &= E\left[\sum_{i=0}^{N}(q_i u_i^2 + a_i(E(\sum_{k=0}^{j-1} b_{i,i-k}\tilde{x}_{i-k} \mid y^i) - \sum_{k=0}^{j-1} b_{i,i-k}\tilde{x}_{i-k})^2)\right] \\
&= E\left[\sum_{i=0}^{N}(q_i \tilde{u}_i^2 + q_i p_i^2 + a_i(E(\sum_{k=0}^{j-1} b_{i,i-k}\tilde{x}_{i-k} \mid y^i) - \sum_{k=0}^{j-1} b_{i,i-k}\tilde{x}_{i-k})^2)\right] \\
&\geqslant E\left[\sum_{i=0}^{N} q_i \tilde{u}_i^2 + a_i(E(\sum_{k=0}^{j-1} b_{i,i-k}\tilde{x}_{i-k} \mid y^i) - \sum_{k=0}^{j-1} b_{i,i-k}\tilde{x}_{i-k})^2)\right].
\end{aligned}$$

We now note that the sigma field generated by $\tilde{y}^i$ is the same as the sigma field generated by y^i where

$$\tilde{y}_i = \tilde{u}_i + w_i$$
$$= v_i - p_i$$

since p_i is y^{i-1} measurable.

Hence,

$$\underset{\gamma_N}{\text{Min}}\ J(h^N,\gamma^N) \geqslant E\left|\sum_{i=0}^{N}(q_i\tilde{u}_i^{\,2} + a_i(E(\sum_{k=0}^{j-1} b_{i,i-k}\tilde{x}_{i-k} \mid \tilde{y}^i) - \sum_{k=0}^{j-1} b_{i,i-k}\tilde{x}_{i-k})^2)\right|$$

and therefore the cost functional may be optimized under the structural constraint (4.21).

4.5. Optimal Linear Strategies for Second-Order ARMA Models

We now concern ourselves with the design of communication and control strategies which simultaneously optimize over the linear class. In order to facilitate an understanding of the concepts and methodology, we restrict ourselves in this section to a study of second-order ARMA models. The general ARMA model will be studied in the next section.

(i) The problem with Hard Power Constraints.

Problem PH^2

$$\underset{h^N,\gamma^N}{\text{Minimize}}\ E[\sum_{i=0}^{N} a_i(v_i - b_{i,i}\tilde{x}_i - b_{i,i-1}\tilde{x}_{i-1})^2]$$

where

$$u_i = h_i(\tilde{x}_i, y^{i-1})$$
$$\tilde{v}_i = \gamma_i(y^i)$$
$$E[u_i^2] = P_i^2$$

and the x_i's are generated via

$$\tilde{x}_1 = \rho_{10}\tilde{x}_0 + m_0$$
$$\tilde{x}_2 = \rho_{21}\tilde{x}_1 + \rho_{20}\tilde{x}_0 + m_1$$
$$\vdots$$
$$\tilde{x}_{i+1} = \rho_{i+1,1}\tilde{x}_i + \rho_{i+1,i-1}\tilde{x}_{i-1} + m_i$$
$$\vdots$$
$$\tilde{x}_{N+1} = \rho_{N+1,N}\tilde{x}_N + \rho_{N+1,N-1}\tilde{x}_{N-1} + m_N \, .$$

□

Notation

Let

$$\Delta_i^j \equiv E[(\tilde{x}_i - E(\tilde{x}_i \mid y^j))^2] \tag{4.22a}$$

$$\Delta_{ik}^j \equiv E[(\tilde{x}_i - E(\tilde{x}_i \mid y^j))(\tilde{x}_k - E(\tilde{x}_k \mid y^j))] \, . \tag{4.22b}$$

Theorem 4.2. The encoding and decoding policies for problem PH^2, which are optimal over the linear class, are given by

$$u_i = \gamma_i(\tilde{x}_i, y^{i-1}) = \lambda_i(\tilde{x}_i - E(\tilde{x}_i \mid y^{i-1})) \tag{4.23}$$

$$\tilde{v}_i = \delta_i(y^i) = b_{i,i}E(\tilde{x}_i \mid y^i) + b_{i,i-1}E(\tilde{x}_{i-1} \mid y^i) \tag{4.24}$$

and the optimal cost is

$$\sum_{i=0}^{N} a_i(b_{i,i}^2\Delta_i^{\,i} + b_{i,i-1}^2\Delta_{i-1}^{\,i} + 2b_{i,i}b_{i,i-1}\Delta_{i,i-1}^{\,i}) \tag{4.25}$$

where

$$E[\tilde{x}_i \mid y^i] = E[\tilde{x}_i \mid y^{i-1}] + E[\tilde{x}_i \mid y_i] \tag{4.26}$$

$$E[\tilde{x}_i \mid y^{i-1}] = \rho_{i,i-1}E[\tilde{x}_{i-1} \mid y^{i-1}] + \rho_{i,i-2}E[\tilde{x}_{i-2} \mid y^{i-1}] \tag{4.27}$$

$$E[\tilde{x}_{i-1} \mid y^i] = E[\tilde{x}_{i-1} \mid y^{i-1}] + E[\tilde{x}_{i-1} \mid y_i] \tag{4.28}$$

$$E[\tilde{x}_i \mid y_i] = \frac{\lambda_i \Delta_i^{\,i-1}}{P_i^{\,2} + \sigma_{w_i}^{\,2}} \, y_i \tag{4.29}$$

$$E[\tilde{x}_{i-1} \mid y_i] = \frac{\lambda_i(\rho_{i,i-1}\Delta_{i-1}^{i-1}+\rho_{i,i-2}\Delta_{i-1,i-2}^{i-1})}{P_i^2+\sigma_{w_i}^2}\, y_i \tag{4.30}$$

where

$$\lambda_i^2 = \frac{P_i^2}{\Delta_i^{i-1}} \tag{4.31}$$

and the Δ's are generated recursively by

$$\Delta_i^i = \Delta_i^{i-1} \frac{\sigma_{w_i}^2}{P_i^2+\sigma_{w_i}^2} \tag{4.32}$$

$$\Delta_i^{i-1} = \rho_{i,i-1}^2\Delta_{i-1}^{i-1} + \rho_{i,i-2}^2\Delta_{i-2}^{i-1} + 2\rho_{i,i-1}\rho_{i,i-2}\Delta_{i-1,i-2}^{i-1} + \sigma_{m_{i-1}}^2 \tag{4.33}$$

$$\Delta_{i-1}^i = \Delta_{i-1}^{i-1} - \frac{P_i^2}{(P_i^2+\sigma_{w_i}^2)}(\rho_{i,i-1}\Delta_{i-1}^{i-1} + \rho_{i,i-2}\Delta_{i-1,i-2}^{i-1})^2 \tag{4.34}$$

$$\Delta_{i,i-1}^i = (\rho_{i,i-1}\Delta_{i-1}^{i-1} + \rho_{i,i-2}\Delta_{i-1,i-2}^{i-1}) \frac{\sigma_{w_i}^2}{P_i^2+\sigma_{w_i}^2} \tag{4.35}$$

with the initial condition

$$\Delta_0^{-1} = \sigma_{x_0}^2 \; ; \; \Delta_{0,-1}^0 = 0 \text{ and } \Delta_{-1}^0 = 0 \, .$$

□

Proof. Expressions (4.23) and (4.24) are immediate (since we know that the optimum u_i is linear in the innovation and the conditional expectation minimizes the mean square error), and (4.25) follows from the definition of Δ. Expressions (4.26) and (4.28) are due to the fact that y_i is independent of y^{i-1} and (4.27) follows from the definition of $\tilde{x}_i$.

Now,

$$y_i = \lambda_i(\tilde{x}_i - E(\tilde{x}_i \mid y^{i-1})) + w_i$$

and

$$E(\tilde{x}_i \mid y_i) = E(\tilde{x}_i - E(\tilde{x}_i \mid y^{i-1}) \mid y_i) ;$$

hence we have (4.29), and (4.31) follows from the hard power constraint on $E[u_i^2]$.

To obtain (4.30) note that for any two zero-mean Gaussian random variables z_1 and z_2, we may write

$$z_1 = \frac{E[z_1 z_2]}{E[z_2^2]} z_2 + n$$

where n is zero-mean, Gaussian, and independent of z_2.

Thus we have

$$\begin{aligned} y_i = {} & \lambda_i \rho_{i,i-1}(\tilde{x}_{i-1} - E(\tilde{x}_{i-1} \mid y^{i-1})) \\ & + \lambda_i \rho_{i,i-2}(\tilde{x}_{i-2} - E(\tilde{x}_{i-2} \mid y^{i-1})) + \lambda_i m_{i-1} + w_i \end{aligned}$$

and writing

$$(\tilde{x}_{i-2} - E[\tilde{x}_{i-2} \mid y^{i-1}]) = \frac{\Delta^{i-1}_{i-1,i-2}}{\Delta^{i-1}_{i-1}} (\tilde{x}_{i-1} - E(\tilde{x}_{i-1} \mid y^{i-1})) + n \ ,$$

(with n independent of $(\tilde{x}_{i-1} - E(\tilde{x}_{i-1} \mid y^{i-1}))$) we have

$$y_i = \lambda_i(\rho_{i,i-1} + \rho_{i,i-2} \frac{\Delta^{i-1}_{i-1,i-2}}{\Delta^{i-1}_{i-1}})(\tilde{x}_{i-1} - E(\tilde{x}_{i-1} \mid y^{i-1})) + n'$$

and therefore

$$E[\tilde{x}_{i-1} \mid y_i] = E[\tilde{x}_{i-1} - E(\tilde{x}_{i-1} \mid y^{i-1}) \mid y_i]$$

is given by (4.30).

What remains to establish now, is the validity of the recursions for the Δ's.

We have

$$\begin{aligned}\Delta_i^{\,i} &= E[(\tilde{x}_i - E(\tilde{x}_i \mid y^i))^2] \\ &= E[(\tilde{x}_i - E(\tilde{x}_i \mid y^{i-1}) - E(\tilde{x}_i \mid y_i))^2] \\ &= \Delta_i^{\,i-1} - E[E(\tilde{x}_i \mid y_i)^2]\end{aligned}$$

and using (4.29), (4.32) follows. Further,

$$\Delta_i^{\,i-1} = E[(\tilde{x}_i - E(\tilde{x}_i \mid y^{i-1}))^2]$$

and using the definition of $\tilde{x}_i$ we have (4.33).

Now

$$\begin{aligned}\Delta_{i-1}^{i} &= E[(\tilde{x}_{i-1} - E(\tilde{x}_{i-1} \mid y^i))^2] \\ &= E[(\tilde{x}_{i-1} - E(\tilde{x}_{i-1} \mid y^{i-1}) - E(\tilde{x}_{i-1} \mid y_i))^2] \\ &= \Delta_{i-1}^{i-1} - E[E(\tilde{x}_{i-1} \mid y_i)^2] \\ &= \Delta_{i-1}^{i-1} - \frac{P_i^{\,2}}{(P_i^{\,2}+\sigma_{w_i}^{\,2})}\,\frac{(\rho_{i,i-1}\Delta_{i-1}^{i-1}+\rho_{i,i-2}\Delta_{i-1,i-2}^{i-1})^2}{\Delta_i^{\,i-1}}\end{aligned}$$

i.e., we have (4.34). Finally,

$$\begin{aligned}\Delta_{i,i-1}^{i} &= E[(\tilde{x}_i - E(\tilde{x}_i \mid y^i))(\tilde{x}_{i-1} - E(\tilde{x}_{i-1} \mid y^i))] \\ &= E[(\tilde{x}_i - E(\tilde{x}_i \mid y^{i-1}) - E(\tilde{x}_i \mid y_i))(\tilde{x}_{i-1} - E(\tilde{x}_{i-1} \mid y^{i-1}) - E(\tilde{x}_{i-1} \mid y_i))] \\ &= \Delta_{i,i-1}^{i-1} - E[E(\tilde{x}_i \mid y_i)E(\tilde{x}_{i-1} \mid y_i)]\end{aligned}$$

and

$$\begin{aligned}\Delta_{i,i-1}^{i-1} &= E[(\tilde{x}_i - E(\tilde{x}_i \mid y^{i-1}))(\tilde{x}_{i-1} - E(\tilde{x}_{i-1} \mid y^{i-1}))] \\ &= \rho_{i,i-1}\Delta_{i-1}^{i-1} + \rho_{i,i-2}\Delta_{i-1,i-2}^{i-1}\,,\end{aligned}$$

along with

$$E[E(\tilde{x}_i \mid y_i)\, E(\tilde{x}_{i-1} \mid y_i)] = \frac{\lambda_i^2 \Delta_i^{i-1}(\rho_{i,i-1}\Delta_{i-1}^{i-1} + \rho_{i,i-2}\Delta_{i-1,i-2}^{i-1})}{P_i^2 + \sigma_{w_i}^2}$$

$$= \frac{P_i^2}{P_i^2 + \sigma_{w_i}^2}\,(\rho_{i,i-1}\Delta_{i-1}^{i-1} + \rho_{i,i-2}\Delta_{i-1,i-2}^{i-1})$$

implies (4.35).

The initial conditions for the above hard constraint version are immediate, and it is a straightforward task to verify that using the given recursions, all expressions required for the solution to PH^2 can be generated.

(ii) The Soft Constraint Version

It follows from the solution to the hard constraint version (as in the case of the first-order ARMA model) that Problem PS^2 may be solved via the following nonlinear optimal control problem.

Problem NL^2

$$\underset{P_0^2,\ldots,P_N^2}{\text{Minimize}} \sum_{i=0}^{N} (q_i P_i^2 + a_i b_{i,i}^2 \Delta_i^i + a_i b_{i,i-1}^2 \Delta_{i-1}^i + 2a_i b_{i,i} b_{i,i-1} \Delta_{i,i-1}^i)$$

where the Δ's are generated recursively by (4.32) through (4.35).

The solution to Problem NL^2 is given by the following dynamic program:

$$W_{N+1}(\Delta_1;\Delta_2;\Delta_3) = 0$$

$$W_i(\Delta_1;\Delta_2;\Delta_3)$$

$$= \underset{P_i^2}{\mathrm{Min}}[q_i P_i^2 + a_i b_{i,i}^2 (\frac{\Delta_1 \sigma_{w_i}^2}{P_i^2 + \sigma_{w_i}^2})$$

$$+ a_i b_{i,i-1}^2 (\Delta_2 - \frac{P_i^2}{(P_i^2 + \sigma_{w_i}^2)} \frac{(\rho_{i,i-1}\Delta_2 + \rho_{i,i-2}\Delta_3)^2}{\Delta_1})$$

$$+ 2 a_i b_{i,i} b_{i,i-1} (\frac{\sigma_{w_i}^2}{P_i^2 + \sigma_{w_i}^2})(\rho_{i,i-1}\Delta_2 + \rho_{i,i-2}\Delta_3)$$

$$+ W_{i+1}(\rho_{i+1,i}^2 \frac{\sigma_{w_i}^2}{P_i^2 + \sigma_{w_i}^2} \Delta_1 + \rho_{i+1,i-1}^2 (\Delta_2 - \frac{P_i^2}{(P_i^2 + \sigma_{w_i}^2)\Delta_1} (\rho_{i,i-1}\Delta_2 + \rho_{i,i-2}\Delta_3)^2)$$

$$+ 2\rho_{i+1,i}\rho_{i+1,i-1}(\rho_{i,i-1}\Delta_2 + \rho_{i,i-2}\Delta_3) \frac{\sigma_{w_i}^2}{P_i^2 + \sigma_{w_i}^2} ;$$

$$\frac{\Delta_1 \sigma_{w_i}^2}{P_i^2 + \sigma_{w_i}^2} ; \frac{(\rho_{i,i-1}\Delta_2 + \rho_{i,i-2}\Delta_3)\sigma_{w_i}^2}{P_i^2 + \sigma_{w_i}^2}) .$$

Further, the optimal value of the cost under linear policies is $W_0(\sigma_{x_0}^2;0;0)$.

□

4.6. Optimal Linear Strategies for the General ARMA Models

(i) The Transformed Problem with Hard Power Constraints

Problem PH^j

$$\underset{h^N,\gamma^N}{\text{Minimize}}\ E[\sum_{i=0}^{N} a_i(\tilde{v}_i - \sum_{k=0}^{j-1} b_{i,i-k}\tilde{x}_{i-k})^2]$$

where

$$u_i = h_i(\tilde{x}_i, y^{i-1})$$
$$\tilde{v}_i = \gamma_i(y^i)$$
$$E[u_i^2] = P_i^2$$
$$y_i = u_i + w_i$$

and

$$\tilde{x}_i = \sum_{k=1}^{j} \rho_{i,i-k}\tilde{x}_{i-k} + m_{i-1} .$$

Theorem 4.3. The encoding and decoding policies for Problem PH^j, which are optimal over the linear class, are given by

$$u_i = \gamma_i(\tilde{x}_i, y^{i-1}) = \lambda_i(\tilde{x}_i - E(\tilde{x}_i \mid y^{i-1})) \tag{4.36}$$

$$\tilde{v}_i = \delta_i(y^i) = \sum_{k=0}^{j-1} b_{i,i-k}E[\tilde{x}_{i-k} \mid y^i] \tag{4.37}$$

and the optimal cost is

$$J^* = \sum_{i=0}^{N} a_i(\sum_{k=0}^{j-1} \sum_{m=0}^{j-1} b_{i,i-k}b_{i,i-m}\Delta^{i}_{i-k,i-m}) \tag{4.38}$$

where

$$E[\tilde{x}_j \mid y^i] = E[\tilde{x}_j \mid y^{i-1}] + E[\tilde{x}_j \mid y_i] \tag{4.39}$$

$$E[\tilde{x}_i \mid y_i] = \frac{\lambda_i \Delta_i^{i-1}}{P_i^2 + \sigma_{w_i}^2} \; y_i \tag{4.40}$$

$$E[\tilde{x}_m \mid y_i] = \frac{\lambda_i \sum_{k=1}^{j} \rho_{i,i-k}\Delta^{i-1}_{i-k,m}}{P_i^2 + \sigma_{w_i}^2} \; y_i \tag{4.41}$$

$$E[\tilde{x}_i \mid y^{i-1}] = \sum_{k=1}^{j} \rho_{i,i-k}E[\tilde{x}_{i-k} \mid y^{i-1}] \tag{4.42}$$

$$\lambda_i^2 = \frac{P_i^2}{\Delta_i^{i-1}} \tag{4.43}$$

and the Δ's are generated recursively by

$$\Delta_i^i = \frac{\Delta_i^{i-1}\sigma_{w_i}^2}{P_i^2+\sigma_{w_i}^2} \tag{4.44}$$

$$\Delta_i^{i-1} = \sigma_{m_{i-1}}^2 + \sum_{k=1}^{j}\sum_{m=1}^{j} \rho_{i,i-k}\rho_{i,i-m}\Delta_{i-k,i-m}^{i-1} \tag{4.45}$$

$$\Delta_m^i = \Delta_m^{i-1} - \frac{P_i^2}{(P_i^2+\sigma_{w_i}^2)\Delta_i^{i-1}}\left(\sum_{k=1}^{j} \rho_{i,i-k}\Delta_{i-k,m}^{i-1}\right)^2 ; (i-j\leqslant m\leqslant i-1) \tag{4.46}$$

$$\Delta_{n,m}^i = \Delta_{n,m}^{i-1} - \frac{P_i^2}{(P_i^2+\sigma_{w_i}^2)}\left(\sum_{k=1}^{j} \rho_{i,i-k}\Delta_{i-k,n}^{i-1}\right)\left(\sum_{k=1}^{j} \rho_{i,i-k}\Delta_{i-k,m}^{i-1}\right) \tag{4.47}$$

$$\Delta_n^{i-1} = \sum_{k=1}^{j}\sum_{p=1}^{j} \rho_{n,n-k}\rho_{n,n-p}\Delta_{n-k,n-p}^{i-1} + \sigma_{m_{n-1}}^2 (i-j\leqslant n\leqslant i-1) \tag{4.48}$$

$$\Delta_{n,m}^{i-1} = \sum_{k=1}^{j} \rho_{n,n-k}\Delta_{n-k,m}^{i-1}\ (i-j\leqslant m<n\leqslant i-1) \tag{4.49}$$

with $\Delta_0^{-1} = \sigma_{x_0}^2$; $\Delta_{k,m}^{-1} = 0$ for either $k<0$ or $m<0$.

Proof. The proof follows in a manner similar to that for the case j=2 discussed in the previous section.

□

(ii) The Transformed Problem with Soft Power Constraints

It follows from the solution to the problem with hard power constraints that Problem PS^j can be solved via the following nonlinear optimal control problem:

$$\underset{P_0^2,\ldots,P_N^2}{\text{Minimize}} \sum_{i=0}^{N} q_i P_i^{\,2} + a_i\Big(\sum_{k=0}^{j-1}\sum_{m=0}^{j-1} b_{i,i-k} b_{i,i-m} \Delta^{\,i}_{i-k,i-m}\Big)$$

where the Δ's are defined via (4.44) through (4.49).

The solution to the above optimal control problem may be obtained via the following dynamic program.

$$W_{N+1} \equiv 0$$

$$W_i(\Delta^{i-1}_{i,i}, \ldots, \Delta^{i-1}_{i,i-(j-1)}, \Delta^{i-1}_{i-1,i-1}, \ldots, \Delta^{i-1}_{i-1,i-(j-1)}, \ldots, \Delta^{i-1}_{i-(j-1),i-(j-1)})$$

$$= \min_{P_i^2} \Bigg[q_i P_i^{\,2} + a_i\Big(\sum_{k=0}^{j-1}\sum_{p=0}^{j-1} b_{i,i-k} b_{i,i-p}\Big(\Delta^{i-1}_{i-k,i-p} -$$

$$\frac{P_i^{\,2}}{\Delta_i^{i-1}(P_i^{\,2}+\sigma^2_{w_i})}\Big(\sum_{n=1}^{j} \rho_{i,i-n}\Delta^{i-1}_{i-n,i-k}\Big)\Big(\sum_{m=1}^{j} \rho_{i,i-m}\Delta^{i-1}_{i-m,i-p}\Big)\Big)$$

$$+ W_{i+1}(\Delta^{i}_{i,i}, \ldots, \Delta^{i}_{i,i-(j-1)}, \Delta^{i}_{i-1}, \ldots, \Delta^{i}_{i-1,i-(j-1)}, \ldots, \Delta^{i}_{i-(j-1),i-(j-1)}) \Bigg]$$

where the relationship between the Δ's is as given in (4.44) through (4.49). The optimal value of the cost is $W_0(\sigma^2_{x_0}, 0, 0, \ldots, 0)$.

(iii) Solutions to the Original Problem

The optimal linear solutions to the original problems PH° and PS° can be generated once the solutions to the transformed problems are available. We indicate how this may be done for the hard constraint version. For the problem with soft power constraints, the procedure is identical, except that we first need to find the optimal power levels by solving the nonlinear optimal control problem via dynamic programming.

Noting that since v_i is y^i measurable, we have

$$x_i - E(x_i \mid y^{i-1}) = \tilde{x}_i - E(\tilde{x}_i \mid y^{i-1})$$

and hence the optimal u_i's are given by

$$u^*_i = \gamma^*_i(x_i, y^{i-1}) = \lambda_i(x_i - E(x_i \mid y^{i-1})) ,$$

with the λ's being the same as those for the transformed version. To obtain the optimum v_i's, note that

$$v^*_i = \delta^*_i(y^i) = \sum_{k=0}^{j-1} b_{i,i-k} E(x_i \mid y^i) ,$$

and recalling that (see Appendix A)

$$x_{i-k} = \tilde{x}_{i-k} - \bar{v}_{i-k-1} ,$$

we have

$$v^*_i = \sum_{k=0}^{j-1} b_{i,i-k} E(\tilde{x}_{i-k} \mid y^i) - \sum_{k=0}^{j-1} \bar{v}_{i-k-1} ,$$

where the $\bar{v}$'s are defined in terms of the preceding v^*'s as in Appendix A and the expected values of the $\tilde{x}$'s are given by (4.39) through (4.42).

4.7. Conclusion

In this chapter we have studied the problem of simultaneously designing communication strategies and control policies for problems involving ARMA models of orders higher than one. For one of the simplest such problems, involving an ARMA model of order 2, we have shown that the optimum linear strategy may be outperformed by an appropriately chosen nonlinear policy. This is done by relating the problem involving the second-order ARMA model, to a problem involving transmission through a Gaussian channel with noisy side information at the decoder. It has further been shown that over the affine class, the optimal strategy consists of transmitting the innovation at each stage.

The solutions which are optimal over the affine class have been studied for the hard power constraint version as well as for the soft power constraint version of the general ARMA model.

CHAPTER 5

THE DECENTRALIZED TWO-PERSON TEAM WITH MULTIPLE INFORMATION CHANNELS

5.1. Introduction

In this chapter we consider decentralized, two-person stochastic team problems, where the action of one agent is transmitted to the other agent through a number of noisy channels simultaneously. This is a generalized version of the decentralized two-person team discussed in Chapter 2, where only a single communication channel was allowed between the two agents.

In Section 5.2 we formalize the problems to be analyzed in this chapter. In Section 5.3 we consider the situation where the channel noises are independent of the input variable to be transmitted, with the observation of the first agent being noise corrupted in general. For this class we show that the optimal strategies are linear and may be found through a related parameter optimization problem. In Section 5.5 we analyze problems where the channel noise is correlated with the input variable to be transmitted and find that the strategies which are optimal over the linear class may be outperformed by non-linear strategies (except for a very restrictive subclass), even when the first agent observes an uncorrupted version of the input. The concluding remarks of Section 5.4 then end this chapter.

5.2. Problem Formulation

We have noted in Chapter 2 that if the first agent observes a noise corrupted version of the variable to be transmitted, and the channel noise is also correlated with this variable, then the linear strategies for the general stochastic team problem may be outper-

formed by nonlinear strategies. In view of this result, we shall restrict ourselves to the following two classes of problems with multiple information channels:

(i) The channel noises are all independent of the input x, while the first agent observes a garbled version of the input x.

(ii) The first agent observes the input x directly, and the channel noises are allowed to be correlated with the input x.

These situations are depicted in Figures 5.1 and 5.2, respectively. We consider the quadratic cost functional $J(\gamma_0,\gamma_1)$ where

$$J(\gamma_0,\gamma_1) = E[k_0u_0^2+s_0u_0x+su_1^2+s_1u_1x]$$
$$u_0 = \gamma_0(\cdot)\ ,\ u_1 = \gamma_1(\cdot)\ ,$$

and the following two classes of problems are thus identified:

Problem P1

$$\underset{\gamma_0,\gamma_1}{\text{Minimize}}\ J(\gamma_0,\gamma_1)$$

where

$$y = \begin{pmatrix} \lambda_1 u_0+w_1 \\ \vdots \\ \lambda_n u_0+w_n \end{pmatrix}$$

$$u_0 = \gamma_0(z)$$
$$u_1 = \gamma_1(y)$$
$$z = x + v$$

and

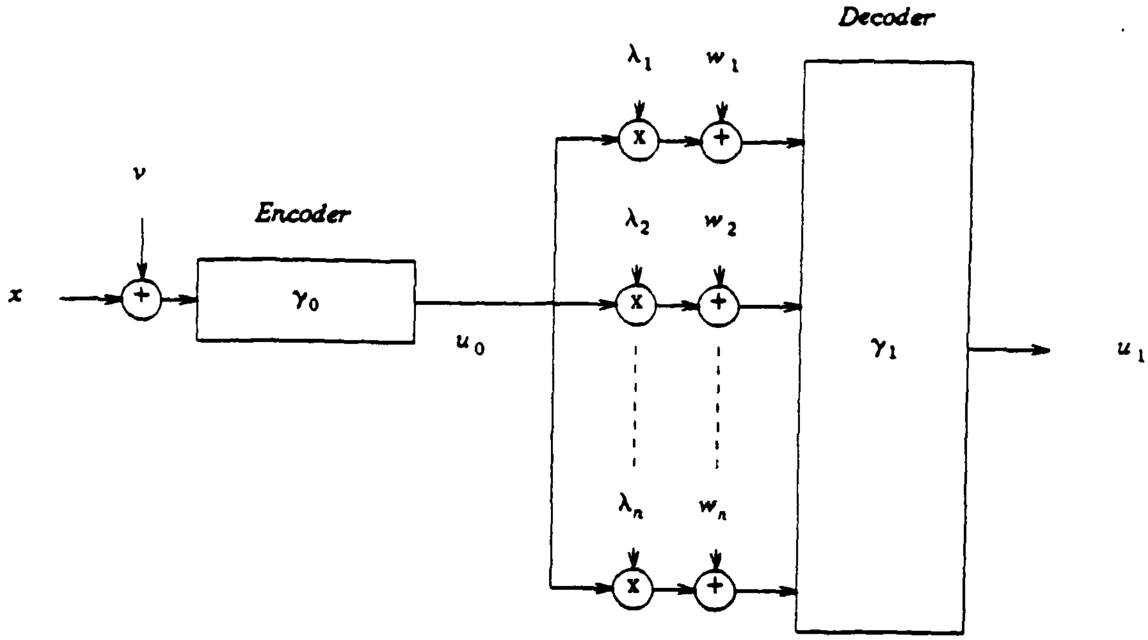

Figure 5.1. The decentralized two-person team, with channel noises independent of each other and of the input.

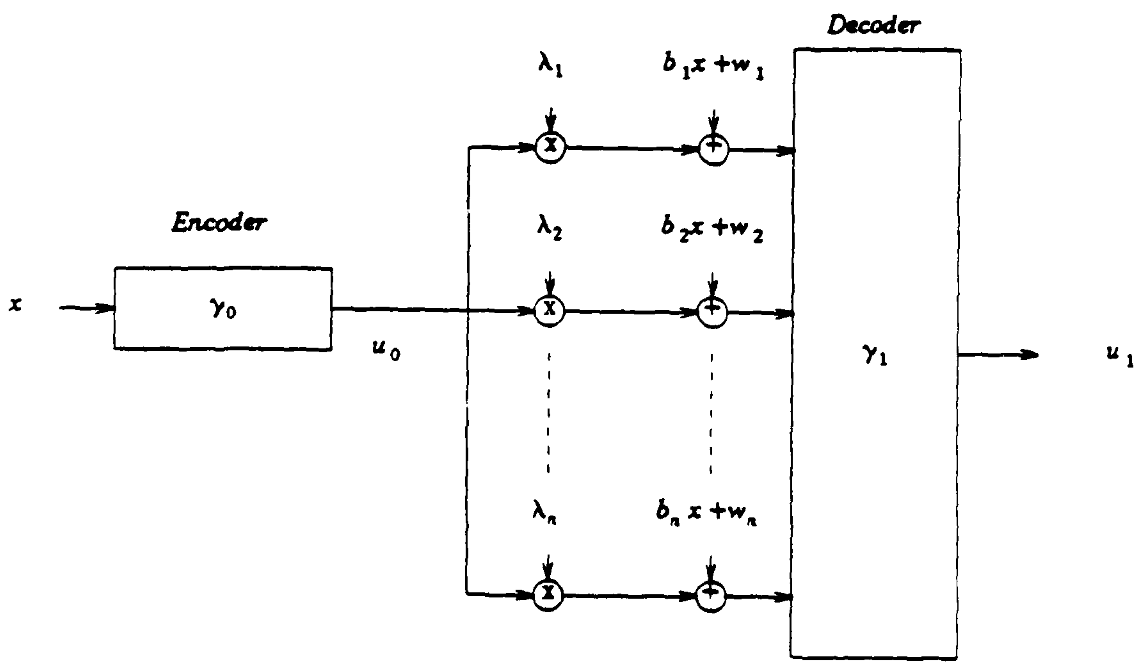

Figure 5.2. The decentralized two-person team with channel noises correlated with the input.

Problem P2

$$\underset{\gamma_0,\gamma_1}{\text{Minimize}}\ J(\gamma_0,\gamma_1)$$

where

$$y = \begin{pmatrix} b_1x+\lambda_1u_0+w_1 \\ b_2x+\lambda_2u_0+w_2 \\ \vdots \\ b_nx+\lambda_nu_0+w_n \end{pmatrix}$$

$$u_0 = \gamma_0(x)$$
$$u_1 = \gamma_1(y)$$

with $x,v,w_1,\ldots,w_n$ being zero mean Gaussian random variables, independent of one another, having variances denoted by $\sigma^2_{(\cdot)}$, and with λ_i's being given positive constants.

We now turn to analyzing the above classes of problems.

5.3. Instances with Optimal Linear Solutions

In this section we analyze Problem P1 formulated in the previous section and show that the optimal strategies are linear. This problem has been studied in Bansal and Basar [1987b] and the analysis is given here for completeness. Towards obtaining the solution to P1 we define

$$m = E[x \mid z]\,,$$

make observations (i) through (iv) as in the case of the single channel problem (Section 2.3 (b)) and obtain the following equivalent problem:

$$\underset{\gamma'_0,\gamma_1}{\text{Minimize}}\ J'(\gamma'_0,\gamma_1)$$

where

$$J'(\gamma'_0,\gamma_1) = E[k_0 u'^2_0 + s_0 u'_0 m + \frac{s_1^2}{4s}(u_1 - m)^2 + c]$$

with

$$u'_0 = \gamma'_0(m)$$
$$u_1 = \gamma_1(y)$$

and c is a constant independent of u'_0 and u_1.

Let J_p denote the infimum of J under the constraint $E[u_o'^2] = P^2$, i.e.,

$$J_p = \inf_{\gamma'_o,\gamma_1;E[\gamma_o'^2]=P^2} J(\gamma'_o,\gamma_1) \ . \qquad (5.1)$$

We first have

$$J_p \geqslant k_o P^2 + \operatorname*{Inf}_{E[\gamma_o'^2]=P^2} s_o E[u'_o m] + \operatorname*{Inf}_{E[\gamma_o'^2]=P^2} E(u_1 - m)^2 + c \ . \qquad (5.2)$$

By Cauchy-Schwartz inequality, we know that

$$\inf_{E[u_o'^2]=P^2} s_o E[u'_o m] = -|s_o| P\sigma_m \ . \qquad (5.3)$$

We now consider the optimization problem $\operatorname*{Inf}_{E[\gamma_o'^2]=P^2} E(u_1 - m)^2$. Since m, $\bar{y}$ and u_1 form a Markov chain, we have

$$I\{m;u_1\} \leqslant I\{m;\bar{y}\} \qquad (5.4)$$

where I{a;b} denotes the mutual information of two random variables a and b.

Also, we have the inequality

$$I\{m;u_1\} \geqslant \frac{1}{2}\log\frac{\sigma_m^2}{E\{(u_1-m)^2\}} \tag{5.5}$$

(Wyner [1970]), and the equality

$$I\{m;\bar{y}\} = H\{\bar{y}\} - H\{\bar{y}\mid m\} \tag{5.6}$$

where $H\{a\}$ is the entropy of the random variable a, and $H\{a \mid b\}$ is the conditional entropy of the random variable a given b.

The correlation matrix for vector $\bar{y}$, with $E[u_o'^2] = P^2$, is

$$\mathbb{C}_{\bar{y}} = \begin{bmatrix} \lambda_1^2P^2+\sigma_{w_1}^2 & \cdots & \lambda_1\lambda_nP^2 \\ \lambda_n\lambda_1P^2 & \cdots & \lambda_n^2P^2+\sigma_{w_n}^2 \end{bmatrix}$$

which has the determinant

$$|\mathbb{C}_{\bar{y}}| = \prod_{i=1}^{n} \sigma_{w_i}^2(1+P^2\sum_i \frac{\lambda_i^2}{\sigma_{w_i}^2}) . \tag{5.7}$$

Since for any n-variate random vector with a fixed mean and a fixed covariance matrix, the maximum entropy is attained by a normal distribution (see, for example, Kagan et al. [1973]), we have

$$H\{\bar{y}\} \leqslant \frac{1}{2}\log(2\pi e)^n |\mathbb{C}_{\bar{z}}| = \frac{1}{2}\log(2\pi e)^n \prod_{i=1}^{n} \sigma_{w_i}^2(1+P^2\lambda) \tag{5.8}$$

where $\lambda \triangleq \sum_i \frac{\lambda_i^2}{\sigma_{w_i}^2}$.

Using (5.8) in (5.6) we get

$$I\{m;\bar{y}\} \leqslant \frac{1}{2}\log(1+P^2\sum_i \frac{\lambda_i^2}{\sigma_{w_i}^2}) \tag{5.9}$$

where we have made use of the expression

$$H\{\bar{y} \mid m\} = \frac{1}{2} \log(2\pi e)^n \prod_{i=1}^{n} \sigma_{w_i}^2 ,$$

since for fixed γ_o', the vector $\bar{y}$ given m consists of n independent Gaussian random variables.

Using (5.5) and (5.9) in (5.4) we get

$$\underset{E[\gamma_o'^2]=P^2}{E\{(u_1-m)^2\}} \geqslant \frac{\sigma_m^2}{1+P^2\lambda} . \tag{5.10}$$

It follows from (5.2) (using (5.3) and (5.10)) that

$$\begin{aligned} J_p &\geqslant k_o P^2 - |s_o| P\sigma_m + \frac{\sigma_m^2}{(1+P^2\lambda)} + c \\ &\geqslant \underset{P\geqslant 0}{\mathrm{Min}}[k_o P^2 - |s_o| P\sigma_m + \frac{\sigma_m^2}{1+P^2\lambda} + c] \\ &= k_o P^{*2} - |s_o| P^*\sigma_m + \frac{\sigma_m^2}{1+P^{*2}\lambda} + c \end{aligned} \tag{5.11}$$

where

$$P^* = \mathrm{Arg}\,\underset{P}{\mathrm{Min}}[k_o P^2 - |s_o| P\sigma_m + \frac{\sigma_m^2}{1+P^2\lambda}] > 0 .$$

Note that P^* necessarily exists, since at $P=0$ the function to be minimized is decreasing and as $P\to\infty$, $J_p\to\infty$, implying that the search can be confined to a closed bounded region of R^1, over which a continuous function always admits a minimum. Taking derivatives, we find

that the value P=P* which attains this minimum satisfies

$$(2k_oP^* - |s_o|\sigma_m)(P^{*2} + \frac{1}{\lambda})^2 = \frac{2P^*\sigma_m^2}{\lambda} \; . \tag{5.12}$$

We now have

$$\begin{aligned} J_{opt} &\triangleq \inf_{\gamma_o,\gamma_1} J(\gamma_o,\gamma_1) = \inf_{P\geq 0} J_p \\ &\geq k_oP^{*2} - (|s_o|\sigma_m)P^* + \frac{\sigma_m^2}{1+P^{*2}\lambda} + c \end{aligned} \tag{5.13}$$

which gives us a lower bound for the infimum of J. The final task is to note that this lower bound is tight and is attained by linear strategies. We thus have Theorem 5.1 below.

Theorem 5.1.

(i) The stochastic team problem P1 formulated in Section 5.2 admits an optimal solution which is linear in the observation variables, and is given by

$$\begin{aligned} \gamma^*_0(z) &= \beta^* z \\ \gamma^*_1(y) &= -\frac{s_1}{2s} E[x \mid y] \\ &= -\frac{s_1}{2s} \frac{\sigma_x^2}{\sigma_x^2+\sigma_v^2} \sum_{i=1}^{n} \frac{\lambda_i\beta^*}{\sigma_{w_i}^2[(1/(\sigma_x^2+\sigma_v^2))+\beta^{*2}\lambda]} y_i \end{aligned}$$

where β^* is given by the solution to the following parameter optimization problem:

$$\beta^* = \arg\min_{\beta} \left[k_0\beta^2(\sigma_x^2+\sigma_v^2) + s_0\beta\sigma_x^2 + \frac{s_1^2}{4s} \cdot \frac{\sigma_x^4}{(\sigma_x^2+\sigma_v^2)(1+\beta^2\lambda(\sigma_x^2+\sigma_v^2))} \right]$$

a solution to which may always be found as discussed earlier.

(ii) The optimal value of the cost is

$$J_{opt} = k_0\beta^{*2}(\sigma_x^2+\sigma_v^2) + s_0\beta^*\sigma_x^2 + \frac{s_1^2\sigma_x^4}{4s(\sigma_x^2+\sigma_v^2)(1+\beta^{*2}\lambda(\sigma_x^2+\sigma_v^2))}$$

$$+ \frac{s_1^2}{4s}\frac{\sigma_x^2\sigma_v^2}{\sigma_x^2+\sigma_v^2} - \frac{s_1^2}{4s}\sigma_x^2 .$$

□

5.4. Nonoptimality of Linear Strategies

In this section we analyze problems where the channel noises are correlated with the input variable x which is to be transmitted through the channel. We show that except for a very restrictive subclass of such problems, the strategies which are optimal over the linear class may be outperformed by appropriately chosen nonlinear strategies.

Since λ_i's are assumed to be nonzero (otherwise the channel is redundant and may be removed) the observation y for Problem P2 is equivalent to the following observation y',

$$y' = \begin{bmatrix} (b_1/\lambda_1)\,x+u_0+(w_1/\lambda_1) \\ \vdots \\ (b_n/\lambda_n)\,x+u_0+(w_n/\lambda_n) \end{bmatrix} .$$

Lemma 5.1. If $b_1/\lambda_1 = b_2/\lambda_2 = \cdots = b_n/\lambda_n$, then the search for optimal strategies for Problem P2 may be confined to the linear class.

Proof. Under the conditions of the lemma we may define

$$u'_0 = u_0 + \frac{b_1}{\lambda_1}x$$

to obtain the equivalent stochastic team problem below:

$$\underset{\gamma'_0,\gamma_1}{\text{Minimize}}\ E[k_0 u'^2_0 + s'_0 u'_0 x + s u_1^2 + s_1 u_1 x]$$

where

$$u'_0 = \gamma'_0(x)$$
$$u_1 = \gamma_1(y)$$

and

$$y = \begin{bmatrix} \lambda_1 u'_0 + w_1 \\ \vdots \\ \lambda_n u'_0 + w_n \end{bmatrix}$$

with

$$s'_0 = s_0 - 2k_0 \frac{b_1}{\lambda_1} .$$

We thus obtain a special case of Problem P1 with $\sigma_v^2 = 0$, for which the optimality of linear strategies has been established in Section 5.3.

□

We now return to the general Problem P2 without the restriction of equality on the b_i/λ_i's, and show that, for this class, nonlinear strategies may outperform the optimal linear strategies.

In the following we shall, for notational convenience, assume that $\lambda_i = 1$ for all i. This does not introduce any loss of generality since a redefinition of the b_i's and the noise variances yield the original problem.

It is sufficient to establish nonoptimality of linear laws for a special two-channel case, which we study next.

Problem P2^2

$$\text{Minimize } J_2(\gamma_0,\gamma_1) = E[k_0 u_0^2+(u_1-x)^2]$$

where

$$u_0 = \gamma_0(x)$$
$$u_1 = \gamma_1(y)$$

and

$$y = \begin{pmatrix} y_1 \\ \\ y_2 \end{pmatrix} = \begin{pmatrix} u_0+bx+w_1 \\ \\ u_0-bx+w_2 \end{pmatrix}$$

with $b>0$ and $\sigma_{w_1}^2 = \sigma_{w_2}^2$.

Lemma 5.2. Problem P2^2 is equivalent to the soft constraint version of the Gaussian Channel Problem with noisy side information at the decoder, studied in Appendix B.

Proof. We need to note that the vector observation y is equivalent to observing

$$\begin{pmatrix} \dfrac{y_1+y_2}{2} \\ \\ \dfrac{y_1-y_2}{2} \end{pmatrix} = \begin{pmatrix} u_0 + \dfrac{w_1+w_2}{2} \\ \\ bx + \dfrac{w_1-w_2}{2} \end{pmatrix}$$

and the lemma is established.

□

Since linear strategies are not optimal for the problem with noisy side information, in view of Lemma 5.2 the linear strategies are not optimal for Problem P2^2.

We next consider the more general two-channel problem with arbitrary (but in view of Lemma 5.1, unequal) b_1 and b_2, but the variances of w_1 and w_2 still being equal.

Problem $P2^{2G}$

$$\text{Minimize } J_2(\gamma_0,\gamma_1) = E[k_0u_0^2 + s_0u_0x + (u_1-x)^2]$$

where

$$u_0 = \gamma_0(x)$$
$$u_1 = \gamma_1(y)$$

and

$$y = \begin{pmatrix} u_0+b_1x+w_1 \\ \\ u_0+b_2x+w_2 \end{pmatrix}$$

with $b_1 > b_2$.

We let

$$u'_0 = u_0 + \frac{b_1+b_2}{2}x$$

to obtain the equivalent cost functional

$$J'_2(\gamma'_0,\gamma_1) = E[k_0u'^2_0 + s'_0u'_0x + (u_1-x)^2]$$

where

$$u'_0 = \gamma'_0(x)$$
$$u_1 = \gamma_1(y)$$

and

$$y = \begin{pmatrix} u_0+bx+w_1 \\ \\ u_0-bx+w_2 \end{pmatrix}$$

with

$$b = \frac{b_1-b_2}{2}$$

and

$$s'_0 = s_0 - k_0(b_1+b_2)\,.$$

The transformed problem may now be viewed as one involving noisy side information at the decoder (cf. Lemma 5.2), and the nonoptimality of linear strategies follows.

5.5. Conclusion

In this chapter we have studied decentralized two-person stochastic team problems with multiple communication channels. For the case when all channel noises are independent of the input variable to be transmitted, we have established the optimality of linear strategies. However, when the channel noise is correlated with the input variable we find that linear strategies are in general not optimal for the case with multiple channels, even when the first agent observes an uncorrupted version of the variable to be transmitted.

The problems in this chapter may be viewed as those involving encoding and transmitting over vector channels with Gaussian noises (under generalized fidelity criteria), where the source output may be distorted prior to encoding or the channel noises may be correlated with the source output.

The classical information transmission problem, viewed as a team problem, generally assumes that one can directly encode the variable to be recovered at the receiving end. A

model which allows distortion prior to transmission was first considered by Dobrushin and Tysbakov [1962]. If we suppose that the message to be transmitted is a temperature read by a digital thermometer or the pixel levels of an image, then the message would not be an exact copy of the object of interest but a noise-bearing variant of it, and the analysis in this chapter would then be applicable.

We also find that for the case where the channel noise is uncorrelated with the encoder input, the nature of the solution is such that it may be considered as first extracting the message from its noisy version under a mean square criterion and then transmitting this extracted message. This is in accordance with the scheme reported for the hard power constraint version with $n=1$ by Wolf and Ziv [1970], as well as with the "disconnection principle" introduced for finite alphabets by Witsenhausen [1980].

We should also note that channels of the form considered in this chapter have been called multipath channels in the literature (Ovseevich and Pinsker [1958]). In such channels, even though there is a single transmitter, the reception is as though a number of channels are operative in parallel. Pinsker [1972] also gives expressions and estimates for the quantity of information contained in observations with respect to an estimated parameter for a fixed and random number of observations. For the special case where each of the λ_i's are assumed to be unity, our result on maximum information between m and $\bar{y}$ (eqn. (3.9)) corresponds to the expression derived in Pinsker [1972], when the number of observations is fixed, and the variables are all Gaussian. Pinsker's approach, however, is based on sufficient statistics, whereas here we have employed known results on entropy maximization.

CHAPTER 6

DECENTRALIZED MULTIPERSON TEAMS

6.1. Introduction

In this chapter we formulate and analyze some decentralized, multiperson, stochastic team problems which are generalizations of the two-person teams studied in Chapter 2. It is assumed that a (possibly noisy) version of a Gaussian random variable, available at one location, is used to generate either a single communication signal or a set of communication signals, which may then be transmitted via noise-corrupted channels to either a single location or to a set of locations.

In Section 6.2 we formulate the problems to be analyzed in this chapter. In Section 6.3 we analyze problems with a single transmitting agent, (synonymously, encoder) and multiple receiving agents (synonymously, decoders). In Section 6.4 we analyze problems with multiple transmitting agents as well as multiple receiving agents. In Section 6.5 we consider problems with multiple transmitting agents and a single receiving agent. The concluding remarks in Section 6.6 then end this chapter.

6.2. Problem Formulation

In this section we formulate three classes of problems involving decentralized multiperson teams, where communication between agents is permitted only via noise-corrupted channels.

First, consider the situation depicted in Figure 6.1, where the measurement variable u_0 is generated based on a (possibly noisy) observation of a Gaussian random variable x, and is then transmitted over a number of noisy channels to various locations. We allow the possibility of controlling the sign of transmission over each of the channels

individually, and the channel noises may, in general, be correlated with the input variable x. All agents are assumed to cooperate in minimizing a quadratic cost functional, and we formally have Problem P1 below:

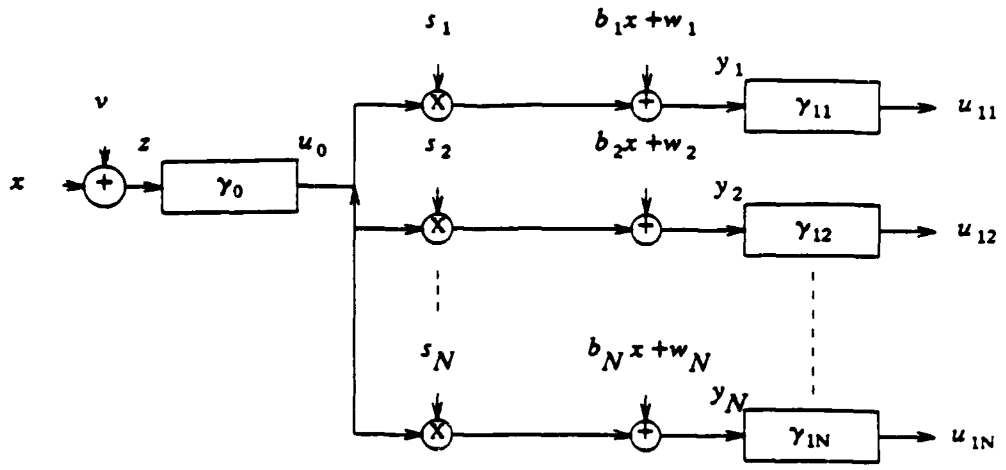

Figure 6.1. The decentralized stochastic team with a single transmitting agent and multiple receiving agents.

Problem P1

$$\underset{\gamma_0;\gamma_{11},\ldots,\gamma_{1N};s_1,\ldots,s_N}{\text{Minimize}} \quad J_1(\gamma_0;\gamma_{11},\ldots,\gamma_{1N};s_1,\ldots,s_N) = E[k_0u_0^2+r_0u_0x+\sum_{i=1}^{N}p_iu_{1i}^2+q_iu_{1i}x]$$

where

$$u_0 = \gamma_0(z)$$
$$u_{1i} = \gamma_{1i}(y_i)$$

with

$$z = x + v$$
$$y_i = s_i u_0 + b_i x + w_i \, .$$

Here $k_0, p_1, \ldots, p_N$ are given positive constants, r_0, q_i's and b_i's are arbitrary constants and x, v, $w_1, \ldots, w_N$ are independent Gaussian random variables, with zero mean and prescribed variances. The s_i's $(i=1,\ldots,N)$ are design variables, constrained to be either +1 or -1; these control the sign of transmission over each of the channels individually.

Note that Problem P1 involves the design of only a single communicating agent, and it is only the sign of transmission on different channels that may be controlled. In case we have complete freedom of designing the communication variable for each of the channels individually, we obtain the situation depicted in Figure 6.2, which leads to Problem P2 below:

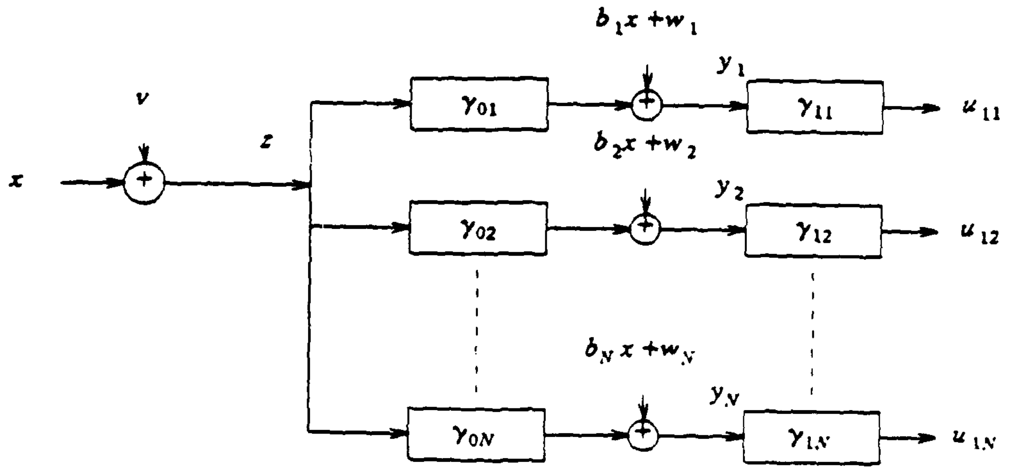

Figure 6.2. The decentralized stochastic team with multiple transmitting as well as multiple receiving agents.

Problem P2

$$\underset{\gamma_{01},\ldots,\gamma_{0N};\gamma_{11},\ldots,\gamma_{1N}}{\text{Minimize}} \quad J_2(\gamma_{01},\ldots,\gamma_{0N};\gamma_{11},\ldots,\gamma_{1N}) = E[\sum_{i=1}^{N} k_i u_{0i}^2 + r_i u_{0i} x + p_i u_{1i}^2 + q_i u_{1i} x]$$

where

$$u_{0i} = \gamma_{0i}(z)$$
$$u_{1i} = \gamma_{1i}(y_i)$$

with

$$z = x + v$$
$$y_i = u_{0i} + b_i + w_i\,.$$

We finally consider the situation shown in Figure 6.3, where multiple communication agents are permitted and noise corrupted versions of all their outputs are made available at a single location. This formally leads to Problem P3 below.

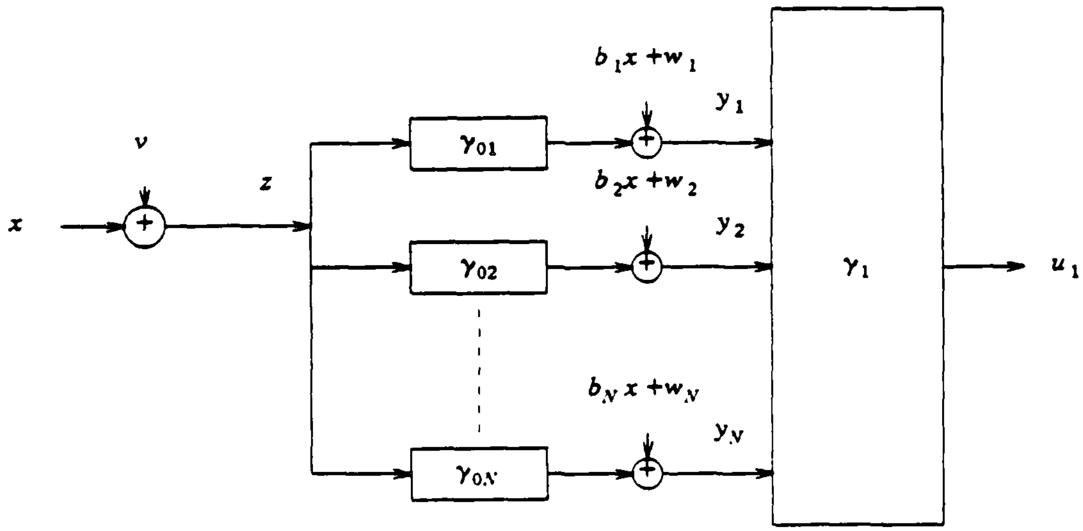

Figure 6.3. The decentralized stochastic team with multiple transmitting agents and a single receiving agent.

Problem P3

$$\text{Minimize } J_3(\gamma_{01}, \ldots, \gamma_{0N}; \gamma_1) = E[\sum_{i=1}^{N}(k_i u_{0i}^2 + r_i u_{0i} x) + p_1 u_1^2 + q_1 u_1 x]$$

where

$$u_{0i} = \gamma_{0i}(z)$$
$$u_1 = \gamma_1(y_1, \ldots, y_N)$$

with

$$z = x + v$$
$$y_i = u_{0i} + b_i x + w_i .$$

The three classes of problems identified in this chapter may all be viewed as generalizations of the decentralized two-person team problem studied in Chapter 2, where it was established that linear strategies are optimal if either one of the following two conditions holds: (a) the channel noise is uncorrelated with the input, or (b) an uncorrupted version of x is available at the encoder. Therefore, in order to study how the results for the two-agent problem carry over to the multiple-agent problems, we shall restrict ourselves to the following two conditions for each of the Problems P1, P2 and P3:

Case (a): $\sigma_v^2 = 0$, b_i's are arbitrary, and

Case (b): $\sigma_v^2 \neq 0$, $b_i = 0$ for all $i=1,\ldots,N$.

In the following three sections we respectively analyze Problems P1, P2 and P3, under conditions (a) and (b) above.

6.3. The Problem with a Single Encoder and Multiple Decoders

In this section we study Problem P1 identified in Section 6.2. By completing squares, we find that the cost function J_1 may be rewritten as

$$J_1(\gamma_0;\gamma_{11},\ldots,\gamma_{1N}) = E[k_0u_0^2+r_0u_0x+\sum_{i=1}^{N}p_i(u_{1i}+\frac{q_i}{2p_i}x)^2-\sum_{i=1}^{N}\frac{q_i^2}{4p_i}\sigma_x^2] \tag{6.1}$$

Case (a): $\sigma_v^2=0$, b_i's are arbitrary.

If $\sigma_v^2=0$, we may define

$$u_{0i}=s_iu_0+b_ix\ , \tag{6.2}$$

and then consider the problem of minimizing

$$E[(u_{1i}+\frac{q_i}{2p_i}x)^2] \tag{6.3}$$

subject to

$$u_{1i}=\gamma_{1i}(u_{0i}+w_i)\ . \tag{6.4}$$

Note that for every fixed $E[u_0^2]=P^2$, we have

$$\begin{aligned} E[u_{0i}^2] &= P^2+b_i^2\sigma_x^2+E[2s_ib_iu_0x] \\ &\leqslant P^2+b_i^2\sigma_x^2+2\,|\,b_iP\sigma_x\,|\ . \end{aligned} \tag{6.5}$$

Since

$$y_i=u_{0i}+w_i \tag{6.6}$$

it follows using (2.10) and (2.11) that

$$I(x;y_i) \leqslant \frac{1}{2} \log \left[\frac{P^2+b_i^2\sigma_x^2+2\,|b_iP\sigma_x|+\sigma_{w_i}^2}{\sigma_{w_i}^2} \right] \tag{6.7}$$

and further, using (2.9), we obtain that

$$E[(u_{1i}+\frac{q_i}{2p_i}x)^2] \geqslant \frac{q_i^2}{4p_i^2} \cdot \frac{\sigma_x^2\sigma_{w_i}^2}{((|P|+|b_i|\sigma_x)^2+\sigma_{w_i}^2)} . \tag{6.8}$$

Let J_{1P} denote the minimum of J_1 under the hard power constraint $E[u_0^2]=P^2$. We then have (with $K \triangleq -\sum_{i=1}^{N}(q_i^2/4p_i)\sigma_x^2$)

$$\begin{aligned} J_{1P} &\geqslant k_0P^2 + \inf_{E[u_0^2]=P^2} E[r_0u_0x] + \inf_{E[u_0^2]=P^2} E[\sum_{i=1}^{N} p_i(u_{1i}+\frac{q_i}{2p_i}x)^2] + K \\ &= k_0P^2 - |r_0P\sigma_x| + \sum_{i=1}^{N}\frac{q_i^2}{4p_i} \cdot \frac{\sigma_x^2\sigma_{w_i}^2}{(|P|+|b_i|\sigma_x)^2+\sigma_{w_i}^2} + K \\ &\geqslant \operatorname*{Min}_{P} \left[k_0P^2 - |r_0P\sigma_x| + \sum_{i=1}^{N}\frac{q_i^2}{4p_i} \cdot \frac{\sigma_x^2\sigma_{w_i}^2}{(|P|+|b_i|\sigma_x)^2+\sigma_{w_i}^2} + K \right] \\ &= k_0P^{*2} - |r_0P^*\sigma_x| + \sum_{i=1}^{N}\frac{q_i^2}{4p_i} \cdot \frac{\sigma_x^2\sigma_{w_i}^2}{(|P^*|+|b_i|\sigma_x)^2+\sigma_{w_i}^2} + K . \end{aligned} \tag{6.9}$$

We thus obtain a lower bound on the optimal cost, and the next task is to note that this lower bound is tight and is achieved by using the linear policies:

$$u^*_0 = \gamma^*_0(x) = -\text{sgn}\, r_0 \cdot \frac{|P^*|}{\sigma_x} x \tag{6.10a}$$

$$u^*_{1i} = \gamma^*_{1i}(y_i) = -\frac{q_i}{2p_i} \frac{s_i(|P^*| + |b_i|\sigma_x)\sigma_x}{((|P^*| + |b_i|\sigma_x)^2 + \sigma_{w_i}^2)} y_i \tag{6.10b}$$

where

$$s_i = \text{sgn}\, b_i \,. \tag{6.10c}$$

We therefore have Theorem 6.1 below.

Theorem 6.1.

(i) The stochastic team problem P1 with $\sigma_v^2 = 0$ admits an optimal solution which is linear in the observation variables, and is given by (6.10), where P^* is given by the solution to the following parameter optimization problem,

$$P^* = \arg\min_P [k_0 P^2 + r_0 P \sigma_x + \sum_{i=1}^{N} \frac{q_i^2}{4p_i} \frac{\sigma_x^2 \sigma_{w_i}^2}{((|P| + |b_i|\sigma_x)^2 + \sigma_{w_i}^2}]$$

which always admits a solution.

(ii) The optimal value of the cost is

$$J^*_1 = k_0 P^{*2} - r_0 P^* \sigma_x + \sum_{i=1}^{N} \frac{q_i^2}{4p_i} \frac{\sigma_x^2 \sigma_{w_i}^2}{((|P^*| + |b_i|\sigma_x)^2 + \sigma_{w_i}^2)} - \sum_{i=1}^{N} \frac{q_i^2}{4p_i} \sigma_x^2$$

□

Case (b): $b_i = 0$ for all i, σ_v^2 is arbitrary.

If b_i's are zero for all i, then all channel noises are independent of the input x. We now define

$$m \triangleq E(x \mid z)$$

and use observations (ii) through (iv) of Section (2.3) to obtain the equivalent problem below:

$$\underset{\gamma'_0, \gamma'_{11}, \ldots, \gamma'_{1N}}{\text{Minimize}} \quad E[k_0 {u'_0}^2 + r_0 u'_0 m + \sum_{i=1}^{N} \frac{q_i^2}{4p_i}(u'_{1i} - m)^2 + K] \tag{6.11}$$

$$u'_0 = \gamma'_0(m)$$
$$u'_{1i} = \gamma'_{1i}(y_i)$$

and K is a constant independent of $u'_0, u'_{11}, \ldots, u'_{1N}$. We thus obtain a special case of the problem discussed in part (a) of this section, and the optimality of linear strategies follows. Note that the s_i's in this case may be chosen arbitrarily, since the b_i's are all zero; hence in the statement of the following theorem we have taken $s_i = +1$ for all i, without any loss of generality.

Theorem 6.2.

(i) The stochastic team problem P1, with $b_i = 0$ for all i, admits an optimal solution which is linear in the observation variables and is given by

$$u^*_0 = \lambda^* z \tag{6.12a}$$

$$u^*_{1i} = -\frac{q_i}{2p_i}\left[\frac{\lambda^* \sigma_x^2}{\lambda^{*2}(\sigma_x^2 + \sigma_v^2) + \sigma_{w_i}^2}\right] y_i \tag{6.12b}$$

where λ^* is given by the solution to the following parameter optimization problem:

$$\lambda^* = \arg\min_{\lambda}[k_0\lambda^2(\sigma_x^2+\sigma_v^2)+r_0\lambda\sigma_x^2 + \sum_{mi=1}^{N}\frac{q_i^2}{4p_i}\frac{\sigma_x^2(\lambda^2\sigma_v^2+\sigma_{w_i}^2)}{(\lambda^2(\sigma_x^2+\sigma_v^2)+\sigma_{w_i}^2)}]$$

a solution to which always exists (Remark 2.1).

(ii) The optimal value for the cost is

$$k_0\lambda^{*2}(\sigma_x^2+\sigma_v^2)+r_0\lambda^*\sigma_x^2 + \sum_{i=1}^{N}\frac{q_i^2}{4p_i}\frac{\sigma_x^2(\lambda^{*2}\sigma_v^2+\sigma_{w_i}^2)}{(\lambda^{*2}(\sigma_x^2+\sigma_v^2)+\sigma_{w_i}^2)} - \sum_{i=1}^{N}\frac{q_i^2}{4p_i}\sigma_x^2 .$$

□

6.4. The Problem with Multiple Encoders and Multiple Decoders

In this section we analyze Problem P2 identified in Section 6.2. First, by completing squares, we find that the cost functional J_2 may be rewritten as

$$J_2(\gamma_{01},\ldots,\gamma_{0N};\gamma_{11},\ldots,\gamma_{1N}) = E[\sum_{i=1}^{N}(k_iu_{0i}^2+r_iu_{0i}x+p_i(u_{1i}+\frac{q_i}{2p_i}x)^2-\frac{q_i^2}{4p_i}\sigma_x^2) . \tag{6.13}$$

Case (a): $\sigma_v^2 = 0$, b_i's are arbitrary.

The analysis is similar to that in Section 2.3. We first define

$$u'_{0i} = u_{0i} + b_ix \tag{6.14}$$

to obtain the equivalent cost functional

$$J'_2(\gamma'_{01},\ldots,\gamma'_{0N};\gamma'_{11},\ldots,\gamma'_{1N}) = E[\sum_{i=1}^{N}k_iu'^2_{0i}+r'_iu'_{0i}x+\frac{q_i^2}{4p_i}(u'_{1i}-x)^2+K]$$

where

$$u'_{0i} = \gamma'_{0i}(x) \tag{6.15a}$$

$$u'_{1i} = \gamma'_{1i}(u'_{0i} + w_i) \tag{6.15b}$$

$$r'_i = r_i - 2k_i b_i \tag{6.15c}$$

and K is a constant independent of $u'_{01},..,u'_{0N},u'_{11},\ldots,u'_{1N}$.

Let J'_{2P} denote the minimum of $J'(\gamma'_0,\gamma'_1)$ under the hard power constraints

$$E[u'^2_{0i}] = P_i^2 . \tag{6.16}$$

We then have

$$\begin{aligned}
J'_{2P} &\geqslant \sum_{i=1}^{N}(k_iP_i^2 + \inf_{E[u'^2_{0i}]=P_i^2} E[r'_i u'_{0i} x] + \inf_{E[u'^2_{0i}]=P_i^2} E[\frac{q_i^2}{4p_i}(u'_{1i}-x)^2]) + K \\
&= \sum_{i=1}^{N}(k_iP_i^2 - |r'_i| P_i\sigma_x + \frac{q_i^2}{4p_i}\frac{\sigma_x^2\sigma_{w_i}^2}{(P_i^2+\sigma_{w_i}^2)}) + K \\
&\geqslant \min_{P_i\geqslant 0}[\sum_{i=1}^{N}(k_iP_i^2 - |r'_i| P_i\sigma_x + \frac{q_i^2}{4p_i}\frac{\sigma_x^2\sigma_{w_i}^2}{(P_i^2+\sigma_{w_i}^2)}) + K] \\
&= \sum_{i=1}^{N}(k_iP^{*2}_i - |r'_i| P^*_i\sigma_x + \frac{q_i^2}{4p_i}\frac{\sigma_x^2\sigma_{w_i}^2}{(P^{*2}_i+\sigma_{w_i}^2)}) + K .
\end{aligned} \tag{6.17}$$

Noting that this bound is tight and is attained by linear policies, we have the following theorem:

Theorem 6.3.

(i) The stochastic team problem P2 with $\sigma_v^2 = 0$ admits an optimal solution which is linear in the observation variables and is given by

$$u^*_{0i} = \gamma^*_{0i}(x) = \lambda^*_i x \tag{6.18a}$$

$$u^*_{1i} = \gamma^*_{1i}(y_i) = -\frac{q_i}{2p_i}\left[\frac{(\lambda^*_i+b_i)\sigma_x^2}{(\lambda^*_i+b_i)^2\sigma_x^2+\sigma_{w_i}^2}\right] y_i \tag{6.18b}$$

where

$$\lambda^*_i = (\frac{P^*_i}{\sigma_x} - b_i) \tag{6.19}$$

and the P^*_i's are found by solving the following parameter optimization problem:

$$P^*_i = \arg\min_{P_i} (k_i P_i^2 + r'_i P_i \sigma_x + \frac{q_i^2}{4p_i} \frac{\sigma_x^2 \sigma_{w_i}^2}{(P_i^2+\sigma_{w_i}^2)}) \tag{6.20}$$

a solution to which always exists (Remark 2.1).

(ii) The optimal value of the cost is

$$J^*_2 = \sum_{i=1}^{N} (k_i P^{*2}_i - r'_i P^*_i \sigma_x + \frac{q_i^2}{4p_i} \frac{\sigma_x^2 \sigma_{w_i}^2}{(P^{*2}_i+\sigma_{w_i}^2)}) + K$$

where

$$K \cdot \sum_{i=1}^{N} (-k_i b_i^2 + r_i b_i - \frac{q_i^2}{4p_i}) \sigma_x^2 .$$

□

Case (b): $b_i = 0$ for all i, σ_v^2 is arbitrary.

If b_i's are all zero, then the channel noises are all independent of the input x. We define

$$m \triangleq E(x \mid z)$$

and use observations (ii) through (iv) of Section 2.3 to obtain the equivalent problem below:

$$\underset{\gamma'_{01},\ldots,\gamma'_{0N},\gamma'_{11},\ldots,\gamma'_{1N}}{\text{Minimize}} \quad E[\sum_{i=1}^{N}(k_i u'^2_{0i} + r_i u'_{0i} m + \frac{q_i^2}{4p_i}(u'_{1i} - m)^2) + K'$$

with

$$u'_{0i} = \gamma'_{0i}(m)$$
$$u'_{1i} = \gamma'_{1i}(y_i)$$

and K' is a constant independent of $u'_{01}, \ldots, u'_{0N}, u'_{11}, \ldots, u'_{1N}$. We thus obtain a special case of the problem discussed in part (a) of this section, and the optimality of linear strategies follows.

Theorem 6.4.

(i) The stochastic team problem P2, with $b_i = 0$ for all i, admits an optimal solution which is linear in the observation variable and is given by

$$u^*_{0i}(z) = \lambda^*_i z \tag{6.21a}$$

$$u^*_{1i}(y_i) = -\frac{q_i}{2p_i}\left[\frac{\lambda^*_i \sigma_x^2}{\lambda^{*2}_i(\sigma_x^2 + \sigma_v^2) + \sigma_{w_i}^2}\right] y_i \tag{6.21b}$$

where λ^*_i is given by the solution to the following parameter optimization problem:

$$\lambda^*_i = \arg\min_{\lambda_i}[k_i\lambda_i^2(\sigma_x^2+\sigma_v^2)+r_i\lambda_i\sigma_x^2+\frac{q_i^2}{4p_i}\frac{\sigma_x^2(\lambda_i^2\sigma_v^2+\sigma_{w_i}^2)}{(\lambda_i^2(\sigma_x^2+\sigma_v^2)+\sigma_{w_i}^2)}]$$

a solution to which always exists (Remark 2.1).

(ii) The optimal value of the cost is

$$\sum_{i=1}^{N}(k_i\lambda^{*2}_i(\sigma_x^2+\sigma_v^2)+r_i\lambda^*_i\sigma_x^2+\frac{q_i^2}{4p_i}\frac{\sigma_x^2(\lambda^{*2}_i\sigma_v^2+\sigma_{w_i}^2)}{(\lambda^{*2}_i(\sigma_x^2+\sigma_v^2)+\sigma_{w_i}^2)}-\frac{q_i^2}{4p_i}\sigma_x^2).$$

□

6.5. The Problem with Multiple Encoders and a Single Decoder

In this section we study Problem P3 identified in Section 6.2. We find that linear strategies are in general nonoptimal for this class of problems even under the restrictions that all channel noises are independent of the input and the encoder observes an uncorrupted version of the input.

In order to see why linear strategies are not optimal, we consider the special case with N=2 and $b_1=b_2=\sigma_v^2=0$, and impose the additional constraints

$$E[u_1^2] \leqslant P_1^2 \tag{6.22a}$$

$$E[u_2^2] \leqslant P_2^2\,. \tag{6.22b}$$

We thus have a problem of constructing real-time encoding and decoding strategies for a communication system with hard power constraints where the source dimension is one and the channel dimension is two. For this problem it is known that linear strategies are not optimal (Shannon [1949], Wozencraft and Jacobs [1965]).

Our task now is to show that it is possible to construct problems with soft power constraints, for which linear strategies are not optimal. This may be done by an analysis

similar to that used in Appendix B. In particular, if we let $k_1=k_2=100.0/(186.0423)^2$, $\sigma_x^2=100.0$, $\sigma_{w_1}^2=\sigma_{w_2}^2=1.0$, then the optimum strategies which are optimal over the linear class yield a cost of 1.07213, whereas nonlinear strategies of the form

$$u_{01} = \gamma_{01}(x) = x \tag{6.23a}$$

$$u_{02} = \gamma_{02}(x) = x - \operatorname{sgn} x \tag{6.23b}$$

$$u_1 = \begin{cases} (y_1+y_2+1)/2 & \text{if } y_1 \geqslant 0 \\ (y_1+y_2-1)/2 & \text{if } y_1 < 0 \end{cases} \tag{6.24}$$

yield a cost of 1.06634.

We now consider Problem P3 under the additional restriction that the policies lie within the linear class. Performance bounds and optimum *linear coding schemes* for discrete-time multichannel communication systems have been investigated by Başar, Sankur and Abut [1980]. They have observed that when the source dimension (n) is less than the channel dimension (m), then the performance of real-time linear encoders is fairly close to the optimum achievable performance found via channel capacity. This does not hold for the case where n>m, where the performance is found to improve significantly by allowing nonlinearities in the encoding policies.

For stochastic team problems of the kind that are of interest here, we have n=1 and m=N. Thus, in view of the results reported by Başar et al. [1980], it is of interest to study optimality over the class of linear encoders, where the loss in performance may be offset by the gain associated with avoiding the complexity of more general encoders. Optimum linear encoding and decoding strategies for the two-stage stochastic team prob-

lem have been studied in Başar [1980], where a rigorous solution to the general problem has been provided. Here we are interested in a special case of that general problem, i.e., the case where the source dimension is one, but with the difference that there are additional product terms in the cost functional, the encoder input is possibly noise-corrupted, and the channel noises may be correlated with the input. We now study optimality over the linear class for this case.

Under the restriction that the encoders use linear policies, we have

$$u_{0i} = \lambda_i(x+v) \tag{6.25}$$

which implies that the observation vector $\bar{y}$ is given by

$$\bar{y} = \begin{bmatrix} \lambda_1(x+v)+b_1x+w_1 \\ \lambda_2(x+v)+b_2x+w_2 \\ \vdots \\ \lambda_N(x+v)+b_Nx+w_N \end{bmatrix}. \tag{6.26}$$

The covariance of x and $\bar{y}$ is given by

$$\mathrm{Cov}(x,\bar{y}) = \begin{bmatrix} \Sigma_{xx} & \Sigma_{x\bar{y}} \\ \Sigma_{x\bar{y}}^T & \Sigma_{\bar{y}\bar{y}} \end{bmatrix} \tag{6.27}$$

where

$$\Sigma_{xx} = \sigma_x^2 \tag{6.28}$$

$$\Sigma_{\bar{y}\bar{y}} = \begin{bmatrix} (\lambda_1+b_1)^2\sigma_x^2+\lambda_1^2\sigma_v^2+\sigma_{w_1}^2 & (\lambda_1+b_1)(\lambda_N+b_N)\sigma_x^2+\lambda_1\lambda_N\sigma_v^2 \\ \vdots & \vdots \\ (\lambda_1+b_1)(\lambda_N+b_N)\sigma_x^2+\lambda_1\lambda_N\sigma_v^2 & (\lambda_N+b_N)^2\sigma_x^2+\lambda_N^2\sigma_v^2+\sigma_{w_N}^2 \end{bmatrix} \tag{6.29}$$

and

$$\Sigma_{x\bar{y}} = [(\lambda_1+b_1)\sigma_x^2, \ldots, (\lambda_N+b_N)\sigma_x^2] \tag{6.30}$$

We now have

$$E(x \mid \bar{y}) = (\Sigma_{x\bar{y}}\Sigma_{\bar{y}\bar{y}}^{-1})\bar{y} \tag{6.31}$$

and the error in estimating x from the vector observation $\bar{y}$ is given by

$$e(\lambda_1, \ldots, \lambda_N) = \Sigma_{xx} - \Sigma_{x\bar{y}}\Sigma_{\bar{y}\bar{y}}^{-1}\Sigma_{x\bar{y}}^{T}. \tag{6.32}$$

The final task is to optimize over the linear coefficients $(\lambda_1, \ldots, \lambda_N)$, and noting that the cost function for P3 may be rewritten as

$$J_3(\gamma_{01}, \ldots, \gamma_{0N};\gamma_1) = E[\sum_{i=1}^{N} k_i u_{0i}^2+r_i u_{0i}x+p_1(u_1+\frac{q_1}{2p_1}x)^2-\frac{q_1^2}{4p_1}\sigma_x^2], \tag{6.33}$$

we formally have Theorem 6.5 below:

Theorem 6.5.

(i) The optimum *linear* strategies for Problem P3 are given by

$$u^*_{0i} = \gamma^*_{0i}(z) = \lambda^* z \tag{6.34a}$$

$$u^*_1 = \gamma^*_1(\bar{y}) = -\frac{q_1}{2p_1}(\Sigma_{x\bar{y}}\Sigma_{\bar{y}\bar{y}}^{-1})\bar{y} \tag{6.34b}$$

where $\sum_{x\bar{y}}$ and $\sum_{\bar{y}\bar{y}}$ are defined by (6.29) and (6.30), with λ_i replaced by λ^*_i, where the λ^*_i's are given by the solution to the following parameter optimization problem:

$$(\lambda^*_1, \ldots, \lambda^*_N) = \arg \min_{(\lambda_1, \ldots, \lambda_N)} \Big[\sum_{i=1}^{N} k_i\lambda_i^2(\sigma_x^2+\sigma_v^2)+r_i\lambda_i\sigma_x^2+\frac{q_1^2}{4p_1} e(\lambda_1, \ldots, \lambda_N)\Big] \tag{6.35}$$

where $e(\lambda_1, \ldots, \lambda_N)$ is defined by (6.32). (The above parameter optimization problem always admits a solution.)

(ii) The optimal value of the cost under linear policies is

$$J^*_3 = E\Big[\sum_{i=1}^{N} k_i\lambda^{*2}_i(\sigma_x^2+\sigma_v^2)+r_i\lambda^*_i\sigma_x^2+\frac{q_1^2}{4p_1}(\sigma_x^2-e(\lambda^*_1, \ldots, \lambda^*_N)) .$$

□

One special case is of particular interest, that with $\sigma_v^2=0$; $b_1=b_2=..=b_N=0$, $r_1=r_2=..=r_N=0$. In this case we get

$$e(\lambda_1, \ldots, \lambda_N) = \frac{\sigma_x^2}{1+\lambda_1^2\dfrac{\sigma_x^2}{\sigma_{w_1}^2}+\ldots+\lambda_N^2\dfrac{\sigma_x^2}{\sigma_{w_N}^2}} \tag{6.36}$$

which implies that the optimum linear coefficients $(\lambda^*_1, \ldots, \lambda^*_N)$ may be found by solving the following parameter optimization problem:

$$\text{Minimize}_{(P_1^2, \ldots, P_N^2)} \; J(P_1^2, \ldots, P_N^2)$$

where

$$J(P_1^2, \ldots, P_N^2) \equiv \sum_{i=1}^{N} k_iP_i^2+\frac{q_1^2}{4p_1}\,\frac{\sigma_x^2}{1+\sum_{i=1}^{N}(P_i^2/\sigma_{w_i}^2)} \tag{6.37}$$

and

$$P_i^2 \equiv \lambda_i^2 \sigma_x^2 . \tag{6.38}$$

The optimum coefficients from this parameter optimization problem may be found by applying Theorem 2.2 of Başar [1980] as follows. We order the channels such that

$$q_1 \sigma_{w_1}^2 \leqslant q_2 \sigma_{w_2}^2 \leqslant .... \leqslant q_N \sigma_{w_N}^2 . \tag{6.39}$$

The derivative of $J(P_1^2, \ldots, P_N^2)$ with respect to P_i^2 is

$$k_i - \frac{q_1^2}{4p_1} \frac{\sigma_x^2 / \sigma_{w_i}^2}{(1+\sum_{i=1}^{N} P_i^2/\sigma_{w_i}^2)} , \tag{6.40}$$

and clearly all first derivatives cannot be zero unless the product $k_i \sigma_{w_i}^2$ is equal for all i.

If this product is indeed equal for all i, then we may arbitrarily choose one channel over which to transmit the information, and all first-order necessary conditions (which are also sufficient) are satisfied.

On the other hand, if the product $k_i \sigma_{w_i}^2$ is not equal for all i, we can choose any channel from the set for which this product is the smallest (this set may be a singleton). For the remaining channels with larger $k_i \sigma_{w_i}^2$ the first derivative is increasing at zero, implying that zero is indeed optimum.

Under the arrangement (6.39), it is therefore sufficient to choose P_1^2 possibly nonzero while all P_i^2 for i=2,...,N may be restricted to be zero, and we get

$$\arg \min_{(P_1^2, \ldots, P_N^2)} J(P_1^2, \ldots, P_N^2) = (P_1^{*2}, \ldots, P_N^{*2})$$

where

$$P^{*2}_{1} = \mathrm{sqrt}(\mathrm{Max}\{0, \frac{\sigma_x \sigma_{w_1}}{k_1^{1/2}} - \sigma^2_{w_1}\}) \tag{6.41}$$

and

$$P^{*2}_{i} = 0 \quad \text{for all } i=2,\dots,N\,.$$

We therefore have the following corollary to Theorem 6.5.

Corollary 6.1.

Consider Problem P3, under the additional restrictions that

$$\sigma_v^2=0;\ b_1=b_2=..=b_N=0;\ r_1=r_2=..=r_N=0\,.$$

If we assume that the channels are ordered such that (6.39) holds, then the optimal linear strategies are given by

$$u^*_{01} = \gamma^*_{01}(x) = \lambda^*_1 x$$
$$u^*_{0i} = \gamma^*_{0i}(x) = 0 \quad \text{for all } i=2,\dots,N$$

and

$$u^*_1 = \gamma^*_1(\bar{y}) = -\frac{q_1}{2p_1} \cdot \frac{\lambda^*_1 \sigma_x^2}{\lambda^{*2}_1 \sigma_x^2 + \sigma^2_{w_1}}\, y_1$$

where

$$\lambda^{*2}_1 = \frac{1}{\sigma_x^2} P^{*2}_1\,,$$

and P^{*2}_1 is defined by (6.41).

(ii) The optimal value of the cost is

$$k_1 P^{*2}_1 + \frac{q_1^2}{4p_1} \frac{\sigma_x^2 \sigma_{w_1}^2}{(P^{*2}_1 + \sigma_{w_1}^2)} - \frac{q_1^2}{4p_1} \sigma_x^2 .$$

□

6.6. Conclusion

In this chapter we have formulated and analyzed three classes of decentralized, multiperson, stochastic team problems, where communication between agents is permitted only via noise-corrupted channels.

For problems involving a single transmitting agent and multiple receiving agents we have shown that the optimum strategies are linear when either the encoder observes an uncorrupted version of the variable to be transmitted, or when all channel noises are independent of the input.

For problems involving multiple transmitting agents and multiple receiving agents we again find that in case the encoders observe an uncorrupted version of the variable to be transmitted, or if all channel noises are independent of the input, then the optimum strategies are linear.

However, for the simplest classes of problems involving multiple transmitting agents and a single receiving agent, we find that the strategies which are optimal over the linear class may be outperformed by appropriately chosen nonlinear policies, and for this case we have provided strategies which are optimal within the linear class.

CHAPTER 7

PROBLEMS WITH INCOMPLETE CHANNEL DESCRIPTION

7.1. Introduction

In this chapter, we expand on the framework developed so far, by allowing incomplete statistical description of the channel used to transmit measurements between the decentralized agents. We still operate under nonclassical information patterns and consider a number of cases depending on whether there are "hard" power constraints or "soft" power constraints on some of the decision variables and/or soft costs on communication. We obtain minimax rules in all cases, some being saddle points and others not, the techniques of derivation being very much case dependent. These are all important prototype problems which could be considered essential building blocks in multistage, distributed decision making under nonclassical information and with partial statistical description.

In particular, we assume that the variable transmitted through the channel is corrupted not only by an independent Gaussian noise of given variance, but also by an unknown channel noise which is only known to satisfy certain power constraints. Further, this unknown noise is allowed to be correlated with either the input or the output of the encoder. (Recall that in Chapter 2 we had allowed part of the channel noise to vary *linearly* with the input).

We seek optimal solutions under a worst case analysis. We may therefore consider the unknown channel noise as being controlled by an adversary or "jammer," who intelligently uses the knowledge of either the input or the output of the channel to design a

jamming strategy. Thus the problems may be viewed as those representing extended versions of the standard Gaussian Test Channel (Gallager [1968]), further including an intelligent jammer.

The organization of this chapter is as follows. In Section 7.2 we identify the problems to be analyzed in the sequel. In particular, three kinds of communication systems are identified, each of which is subsequently analyzed under a variety of fidelity criteria, which are also defined in Section 7.2. The three types of communication systems are analyzed in Sections 7.3, 7.4 and 7.5. The concluding remarks in Section 7.6 then end this chapter.

In what follows, we shall refer to the problems studied in this chapter alternatively as zero-sum games, since the jammer wishes to maximize the same criterion which the encoder-decoder pair is trying to minimize. A general discussion on zero sum games may be found in Başar and Olsder [1982].

7.2. Problem Description

7.2.1. Channel description

Following the formulation of Başar [1983], consider the communication systems depicted in Figures 7.1, 7.2 and 7.3 which represent extended versions of the standard Gaussian Test Channel, (Gallager [1968]), and include an intelligent jammer who has access either to the input or to the output of the encoder. The input to the encoder is a Gaussian random variable with zero mean and unit variance, denoted $u \sim N(0,1)$. The transmitter encodes the input signal u into a variable $y = \gamma(u)$ with the encoding policy γ being an

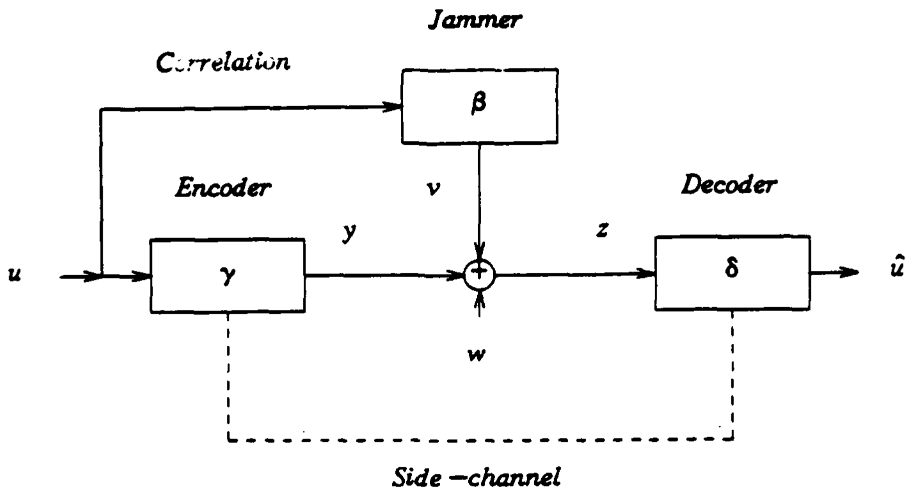

Figure 7.1. Schematics for games of Type 1.

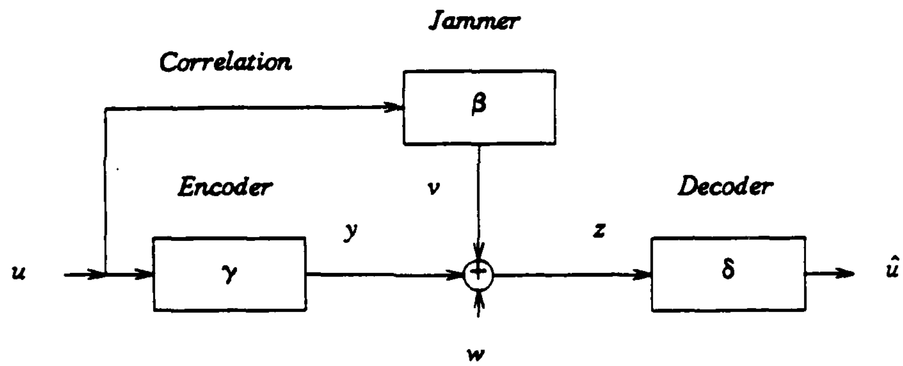

Figure 7.2. Schematics for games of Type 2.

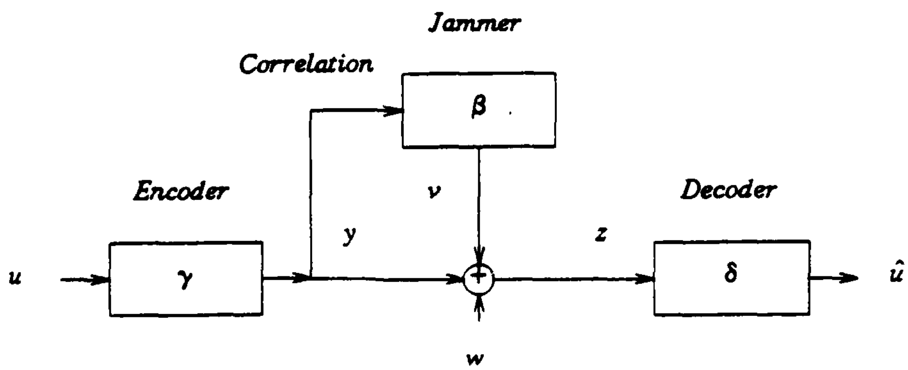

Figure 7.3. Schematics for games of Type 3.

element of the space Γ_e of random mappings or an element of the space Γ_{ed} of deterministic mappings. The encoder is further restricted to satisfy certain power constraints as to be elucidated in the sequel. When the encoder map is allowed to be probabilistic we refer to the problem as a game of Type 1, in which case we also allow for a side channel that permits the transmission of information regarding the structure of the probabilistic encoder to the receiver (Figure 7.1). When the encoder map is necessarily deterministic it is a game of Type 2, for which the side channel is superfluous (Figure 7.2).

In games of Types 1 and 2 the jammer is allowed to observe the input u. The jammer then feeds back into the channel a random variable $v = \beta(u)$ so that the input to the decoder is

$$z = \gamma(u) + \beta(u) + w \ .$$

The jammers policy is an element in the space Γ_j of random mappings also restricted to satisfy certain power constraints.

At the receiver, the decoder $\delta(z)$ is chosen so as to cooperate with the encoder in minimizing a (generalized) quadratic distortion measure $J(\gamma,\delta,\beta)$. Here the noise w is assumed to be Gaussian with mean zero, variance σ_w^2, and is independent of u.

Figure 7.3 represents the situation when the jammer has access to the output of the encoder instead of u. This situation shall be referred to as a game of Type 3. Games of Type 3 are investigated in Bansal and Başar [1989] under a variety of fidelity criteria which we next describe.

7.2.2. Fidelity criteria

The general communication game addresses the reproduction of the input u, with high fidelity, at the output of the decoder. This corresponds to minimizing the mean

square error $E\{(\delta(z)-u)^2\}$. In order that this minimization be well defined we restrict ourselves to using encoding and jamming policies which satisfy $E[\gamma^2(u)] \leqslant P_0^2$ and $E[v^2] \leqslant P_J^2$. These are the *hard* power constraints generally assumed in the communication literature, which lead to criterion C_1 below.

Criterion C_1:

$$J_1(\gamma,\delta,\beta) = E\{(\delta(z)-u)^2\} \tag{7.1a}$$

and

$$E[\gamma^2(u)] \leqslant P_0^2\,,\; E[v^2] \leqslant P_J^2\,. \tag{7.1b}$$

This is precisely the criterion adopted in Başar [1983] and Başar and Wu [1985], under which a complete characterization of minimax encoder-decoder policies for games of Types 1 and 2 has been provided in Başar and Wu [1985]. We introduce here some classes of problems which have heretofore received less attention. These are problems where the power levels are not fixed a priori, but are determined as a result of the underlying optimization problem. This is particularly important in situations where it may be possible, for instance, to transmit a larger power at the encoder (at some cost) in order to further decrease the mean square error at the decoder. Mathematically this may be represented as a power constraint which is "implied" or "*soft*" and appears as an additional term in the cost functional. Based on the consideration of tradeoffs between power used and the magnitude of the mean square error term, we have criteria C_2, C_3 and C_4 below where coefficient terms k_0 and α are positive scalars in all cases.

Criterion C_2:

$$J_2(\gamma,\delta,\beta) = E\{k_0\gamma^2(u) + (\delta(z) - u)^2 - \alpha v^2\} \tag{7.2}$$

Criterion C_3:

$$J_3(\gamma,\delta,\beta) = E\{(\delta(z) - u)^2 - \alpha v^2\} \tag{7.3a}$$

$$E[\gamma^2(u)] \leqslant P_0^2 \tag{7.3b}$$

Criterion C_4:

$$J_4(\gamma,\delta,\beta) = E\{k_0\gamma^2(u) + (\delta(z) - u)^2\} \tag{7.4a}$$

$$E[v^2(u)] \leqslant P_J^2 \,. \tag{7.4b}$$

Under the different criteria above, the spaces in which the policies need to be restricted are characterized by using a superscript. For example, Γ_{ed}^{k} denotes the class of deterministic encoder policies which are permissible when criterion C_k is under consideration.

7.2.3. Solution concepts

We are faced with multiperson decision problems with conflicting objectives. We thus seek to obtain worst case solutions by minimax and maximin approaches, and further, saddle-point solutions if they exist.

Under the minimax approach the encoder-decoder pair is assumed to be careful and defensive, wishing to protect against any irrational behavior of the jammer. Thus, under the minimax criterion, we evaluate the upper value $\bar{J}$ of the zero sum game with kernel J,

$$\begin{aligned} \bar{J} = J(\gamma^*,\delta^*,\beta^*_{(\gamma^*,\delta^*)}) &= \min_{\gamma,\delta} \max_{\beta} J(\gamma,\delta,\beta) \\ &= \min_{\gamma,\delta} J(\gamma,\delta,\beta^*_{(\gamma,\delta)}) \end{aligned} \tag{7.5}$$

where

$$\beta^*_{(\gamma,\delta)} = \arg\max_{\beta} J(\gamma,\delta,\beta) \qquad (7.6)$$

assuming that such a solution exists. The triple $J(\gamma^*,\delta^*,\beta^*_{(\gamma^*,\delta^*)})$ defined by (7.5) and (7.6) above is called the *minimax solution* for the corresponding game.

To obtain the maximin solution, we consider instead the lower value of the zero sum game:

$$\begin{aligned} \underline{J} = J(\gamma^*_{\beta^*},\delta^*_{\beta^*},\beta^*) &= \max_{\beta} \min_{(\gamma,\delta)} J(\gamma,\delta,\beta) \\ &= \max_{\beta} J(\gamma^*_{\beta},\delta^*_{\beta},\beta) \end{aligned} \qquad (7.7)$$

where the pair $(\gamma^*_{\beta},\delta^*_{\beta})$ is determined by

$$(\gamma^*_{\beta},\delta^*_{\beta}) = \arg\min_{(\gamma,\delta)} J(\gamma,\delta,\beta)\,. \qquad (7.8)$$

The triple $(\gamma^*_{\beta^*},\delta^*_{\beta^*},\beta^*)$ defined by (7.7) and (7.8) above is the *maximin solution* for the corresponding game.

A saddle-point solution $(\gamma^*,\delta^*,\beta^*)$ will exist if and only if

$$\underline{J} = \bar{J} = J(\gamma^*,\delta^*,\beta^*)\,. \qquad (7.9)$$

A detailed study of the concepts of minimax and maximin strategies and saddle-point solutions may be found in Başar and Olsder [1982].

7.2.4. Notation

We shall use a double subscript notation and refer to a game as Game G_{ik}, with $i \in \{1,2,3\}$ and $k \in \{1,2,3,4\}$. The first subscript i refers to the game type (Section 7.2.1) while the second subscript k indicates that fidelity criterion C_k is under consideration.

with the strategy spaces being Γ_{ed}^{k} or Γ_{e}^{k}, Γ_{r}^{k} and Γ_{j}^{k} for the encoder, decoder and jammer, respectively. Thus for i=3 the jammer has access to the output of the encoder, whereas for i=1 or 2 the jammer taps the input u. Further, for i=1 the encoder is allowed to use probabilistic mappings whereas for i=2, the encoder is necessarily deterministic.

7.2.5. Contributions of the chapter

The existence of a saddle-point solution for game G_{31} is proved in Başar [1983] in a more general framework where it is shown that the parameter space can be partitioned into regions with the saddle point depending *structurally* upon the region of operation. Games G_{i1} for i=1,2 are considered in Başar and Wu [1985] where the existence of a saddle-point solution for G_{11} is established and a complete characterization of minimax and maximin strategies for G_{21} is provided.

In this chapter we study the remaining nine games G_{ik}, $i \in \{1,2,3\}$ and $k \in \{2,3,4\}$. In Section 7.2 we provide saddle-point solutions for games G_{12}, G_{13} and G_{14}. In Section 7.4 we analyze games G_{32}, G_{33} and G_{34}. For games G_{22}, G_{23} and G_{24} saddle-point solutions do not exist; we provide both maximin and minimax solutions for these games in Section 7.5.

7.3. Saddle-Point Solutions for Games G_{12}, G_{13} and G_{14}

Consider the problem described in Section 7.1, with the encoder allowed to use probabilistic mappings (Figure 7.1), under Criteria C2, C3 and C4, respectively. The solution is provided in Theorem 7.1 after introducing some notation.

Preliminary Notation for Theorem 7.1

We introduce scalar parameters P^*_0 and P^*_J by dividing the parameter space into distinct regions as follows:

a. Game G_{12}

Define Regions R_{12}^1, R_{12}^2 and R_{12}^3 by

$$R_{12}^1 : \frac{1}{k_0} \leqslant \sigma_w^2 \tag{7.10a}$$

$$R_{12}^2 : \frac{k_0}{(k_0+\alpha)^2} \leqslant \sigma_w^2 < \frac{1}{k_0} \tag{7.10b}$$

$$R_{12}^3 : \sigma_w^2 < \frac{k_0}{(k_0+\alpha)^2} \,. \tag{7.10c}$$

Now define P^*_0 and P^*_J by

$$P^*_0 = \begin{cases} 0 & \text{in } R_{12}^1 \\ \left(\dfrac{\sigma_w}{k_0^{1/2}} - \sigma_w^2\right)^{1/2} & \text{in } R_{12}^2 \\ \dfrac{\alpha^{1/2}}{(k_0+\alpha)} & \text{in } R_{12}^3 \end{cases} \tag{7.11a}$$

$$P^*_J = \begin{cases} 0 & \text{in } R_{12}^1 \text{ and } R_{12}^2 \\ \left(\dfrac{k_0}{(k_0+\alpha)^2} - \sigma_w^2\right)^{1/2} & \text{in } R_{12}^3 \end{cases} \tag{7.11b}$$

b. Game G_{13}

Define Regions R_{13}^1 and R_{13}^2 by

$$R_{13}^1 : \alpha \geqslant P_0^2/(P_0^2 + \sigma_w^2)^2 \tag{7.12a}$$

$$R_{13}^2 : \alpha < P_0^2/(P_0^2 + \sigma_w^2)^2 . \tag{7.12b}$$

Now define P^*_0 and P^*_J by

$$P^*_0 = P_0 \qquad \text{in } R_{13}^1 \text{ and } R_{13}^2 \tag{7.13a}$$

$$P^*_J = \begin{cases} 0 & \text{in } R_{13}^1 \\ (P_0/\alpha^{1/2}) - (P_0^2+\sigma_w^2) & \text{in} R_{13}^2 \end{cases} . \tag{7.13b}$$

c. Game G_{14}

Define Regions R_{14}^1 and R_{14}^2 by

$$R_{14}^1 : k_0 \geqslant (P_J^2 + \sigma_w^2)^{-1} \tag{7.14a}$$

$$R_{14}^1 : k_0 < (P_J^2 + \sigma_w^2)^{-1} . \tag{7.14b}$$

Now define P^*_0 and P^*_J by

$$P^*_0 = \begin{cases} 0 & \text{in} R_{14}^1 \\ (((P_J^2+\sigma_w^2)^{1/2}/k_0^{1/2})-(P_J^2+\sigma_w^2))^{1/2} & \text{in} R_{14}^2 \end{cases} \tag{7.15a}$$

$$P^*_J = P_J \qquad \text{in } R_{14}^1 \text{ and } R_{14}^2 . \tag{7.15b}$$

Theorem 7.1: Consider the games G_{12}, G_{13} and G_{14} described in Section 1. For each of these games there exists a saddle-point solution $(\gamma^*,\delta^*,\beta^*)$ given as follows:

$$(\gamma^*,\delta^*) = \begin{cases} (P^*_0 u, (P^*_0/(P^{*2}_0+\sigma_w^2+P^{*2}_J))z) & \text{w.p. } 0.5 \\ \\ (-P^*_0 u, (-P^*_0/(P^{*2}_0+\sigma_w^2+P^{*2}_J))z) & \text{w.p. } 0.5 \end{cases} \quad (7.16)$$

$$\beta^*(u) = \eta \quad (7.17)$$

where η is a zero mean Gaussian random variable with variance P^{*2}_J, which is independent of u and w.

Proof: We need to prove that the solutions given above satisfy the saddle-point inequalities:

$$J(\gamma^*,\delta^*,\beta) \leqslant J(\gamma^*,\delta^*,\beta^*) \leqslant J(\gamma,\delta,\beta^*) \quad (7.18)$$

for all permissible triplets (γ,δ,β) and under the stipulation that the side channel is used to carry structural information concerning the probabilistic encoder mappings. The proof is provided by establishing separately the validity of the right-hand side (RHS) and left-hand side (LHS) inequalities of (18) under each criterion for each of the regions in consideration.

(a) Game G_{12}

Region R^1_{12}

(i) The RHS inequality

The problem faced by the encoder-decoder pair, with the jammer's policy fixed as indicated is

$$\min_{\gamma,\delta} E[k_0\gamma^2(u) + (\delta(z)-u)^2]$$

where $z = \gamma(u) + w$.

For each fixed power level ($E[\gamma^2(u)] = P_0^2$) the problem above is the standard Gaussian test channel for which the optimal solution is known to be linear. The optimal power levels may be found by solving the parameter optimization problem:

$$\min_{P_0} [k_0 P_0^2 + \sigma_w^2/(P_0^2 + \sigma_w^2)]$$

which gives $P*_0^2 = 0$ as in the statement of the theorem.

(ii) The LHS inequality

With the encoding-decoding policy fixed at zero, the problem faced by the jammer is

$$\max_{\beta(u)} (-\alpha\beta^2(u) + 1)$$

for which the optimal choice is $\beta*(u) = 0$.

Region R_{12}^2

(i) The RHS inequality

As in the case of Region R_{12}^1, we find that the optimal encoding-decoding policy is linear and the linear encoding coefficient may be found by solving the parameter optimization problem

$$\min_{P_0} [k_0 P_0^2 + \sigma_w^2/(P_0^2 + \sigma_w^2)]$$

which gives $P*_0^2 = (\sigma_w / k_0^{1/2}) - \sigma_w^2$ as in the statement of the theorem.

(ii) The LHS inequality

With the encoding-decoding policy fixed as given, the cost to be maximized by the jammer is obtained by unconditioning the conditional value of the cost incurred for each

realization of the equiprobable random variable which decides the sign of the linear encoding. In the expression thus obtained for the cost to be maximized, the jammer's policy only enters as a term containing $E[\beta^2(u)]$ with the coefficient

$$(((\sigma_w/k_0^{1/2}) - \sigma_w^2)k_0/\sigma_w^2) - \alpha \,.$$

This coefficient is nonpositive in R_{12}^2, implying that $\beta(u) = 0$ is the jammer's optimal policy.

Region R_{12}^3

(i) The RHS inequality

When the jammer uses independent Gaussian noise with the indicated variance, the problem faced by the encoder-decoder pair is

$$\min_{\gamma,\delta} E[k_0\gamma^2(u) + (\delta(z)-u)^2 + K'] \quad ,$$

where $z = \gamma(u)+w+\eta$ and K' is independent of the encoding-decoding policy. For each fixed power level, $E[\gamma^2(u)] = P_0^2$, the problem is the standard Gaussian test channel problem, and thus the coefficient of the optimal linear solution may be found by solving the parameter optimization problem

$$\min_{P_0} [k_0P_0^2 + (k_0/(k_0+\alpha)^2)/(P_0^2+(k_0/(k_0+\alpha)^2))]$$

which gives P^*_0 as in the statement of the theorem.

(ii) The LHS inequality

When the encoder-decoder policy is fixed as indicated, the cost function to be maximized by the jammer is obtained by unconditioning the conditional value of the cost which

gives an expression independent of β, and thus the jamming policy may be chosen as in the statement of the theorem.

(b) Game G_{13}

Region R_{13}^{1}

(i) The RHS inequality

With the jammer's policy fixed at zero, the encoder-decoder pair faces the standard Gaussian test channel problem for which the optimal linear solution is as indicated in the statement of the theorem.

(ii) The LHS inequality

When the encoder-decoder policy is fixed as indicated, the cost function to be maximized by the jammer is obtained by unconditioning the conditional value of the cost which gives

$$E[(-\alpha + P_0^2/(P_0^2+\sigma_w^2)^2)\beta^2(u)+K'']$$

where K'' is independent of β. In the region under consideration the coefficient of $E[\beta^2(u)]$ is negative implying that $\beta(u)=0$ is an optimum.

Region R_{13}^{2}

(i) The RHS inequality

With the jammer's policy fixed as given, the encoder-decoder pair faces the standard Gaussian test channel problem, and the solution is as given.

(ii) The LHS inequality

With the encoding-decoding policy fixed as indicated, the cost function to be maximized by the jammer, obtained by unconditioning the cost, becomes independent of $\beta(u)$ which may, therefore, be chosen as in the statement of the theorem.

(c) Game G_{14}

Region R_{14}^1

(i) The RHS inequality

With the jamming policy fixed as Gaussian noise, the problem faced by the encoder-decoder pair is

$$\min_{\gamma,\delta} E[k_0\gamma^2(u) + (\delta(z)-u)^2]$$

with $z = \gamma(u)+w+\eta$.

The coefficient of the optimal linear solution to the above problem may be found by solving the parameter optimization problem

$$\min_{P_0} k_0P_0^2 + \frac{\sigma_w^2+P_J^2}{P_0^2+P_J^2+\sigma_w^2}$$

for which $P{*}_0^2 = 0$ in the region under consideration.

(ii) The LHS inequality

With the encoder-decoder policy fixed at zero, the cost function is independent of $\beta(u)$ which may be chosen to be Gaussian noise as in the statement of the theorem.

Region R_{14}^2

(i) The RHS inequality

The problem faced by the encoder-decoder pair, with the jammer's policy fixed as Gaussian noise is

$$\min_{\gamma,\delta} E[k_0\gamma^2(u) + (\delta(z)-u)^2]$$

where $z = \gamma(u)+w+\eta$.

The coefficients for the optimal linear solution to the above problem may be obtained by solving the parameter optimization problem

$$\min_{P_0} k_0P_0^2 + (\sigma_w^2+P_J^2)/(P_0^2+P_J^2+\sigma_w^2)$$

which gives the optimal encoding-decoding policies as in the statement of the theorem.

(ii) The LHS inequality

With the encoding-decoding policy fixed as indicated, the cost to be maximized by the jammer, obtained by unconditioning, increases linearly with $E[\beta^2(u)]$. Thus the jammer may choose any random variable with second moment equal to P_J^2; i.e., $\beta(u)$ may be chosen to be independent Gaussian noise as in the statement of the theorem.

□

7.4. Analysis of Games G_{32}, G_{33} and G_{34}

7.4.1. Some useful results

The most relevant paper to this section is Başar [1983], where a game of Type 3 with fidelity Criterion C1 has been studied. Here we reproduce the main theorem of that paper (in the current less general framework) for future reference. First note that if $P_o \leqslant P_J$ the solution is trivial since the jammer can cancel out the signal component. If, however, $P_o > P_J$, then it is convenient to partition the parameter space into two regions

$$\text{R1: } P_o > P_J \text{ and } P_oP_J - P_J^2 < \sigma_w^2$$

$$\text{R2: } P_o > P_J \text{ and } P_oP_J - P_J^2 \geqslant \sigma_w^2$$

and to introduce

$$\lambda^* = -P_J/P_o$$

and

$$t^* = 1 - (P_J^2 + \sigma_w^2)^2/P_o^2P_J^2 \quad .$$

Lemma 7.1. Game G_{31} admits two saddle-point solutions $(\gamma^*,\delta^*,\mu^*)$ and $(-\gamma^*,-\delta^*,\mu^*)$, where

(i) $\gamma^*(u) = P_o u$

(ii) μ^* is the Gaussian probability measure associated with the random variable

$$v = \beta^*(y) = \begin{cases} \lambda^* y & \text{in R1} \\ \\ \lambda^*(1-t^*)^{1/2} y + \eta^* & \text{in R2} \end{cases}$$

where $\eta^* \sim (0, t^*P_J^2)$

(iii) δ^* is the Bayes estimator for u under the least favorable distribution μ^*, computed as

$$\delta^*(z) = \begin{cases} (P_o(1+\lambda^*)/[P_o^2(1+\lambda^*)^2+\sigma_w^2])\, z & \text{in R1} \\ \\ (1/P_o)\, z & \text{in R2} \end{cases} .$$

Proof. See Başar [1983].

In order to establish similar results under Criteria C2, C3 and C4, we shall need the following lemmata:

Lemma 7.2. Consider the stochastic team problem with nonclassical information (which can be obtained from C2 by setting $\beta(\cdot) \equiv 0$)

$$\min_{\gamma,\delta} E\{k_o\gamma^2(u) + (\delta(z) - u)^2\}$$

where $z = \gamma(u)+w$. The optimal solution to this problem is linear, i.e.,

$$\gamma^*(u) = P^*_o u$$

and

$$\delta^*(z) = E[u \mid P^*_o u+w] = [P^*_o/(P_o^{*2}+\sigma_w^2)]\ z \quad ,$$

where

$$P^*_o \triangleq \arg\min_{P} \{k_o P^2 + \sigma_w^2/(P^2+\sigma_w^2)\} .$$

Proof. See Bansal and Başar [1987a].

Lemma 7.3. Consider the stochastic team decision problem with nonclassical information (which can be obtained from C3 by setting $\beta(\cdot) \equiv 0$)

$$\min_{\gamma,\delta} E\{(\delta(z)-u)^2\}$$

where $z=\gamma(u)+w$ and $E[\gamma^2(u)] \leqslant P^2$. The optimal solution is linear and

$$\gamma^*(u) = Pu \quad ,$$

$$\delta^*(z) = [P/(P^2+\sigma_w^2)]\ z \quad .$$

Proof. The problem may be viewed as the standard Gaussian Test Channel problem, from which the optimality of the linear solution follows. Alternatively, the proof follows in a straightforward fashion using a contradiction on Lemma 7.2.

□

We note here that because of the complex structure of the solution to the hard constraint problem (discussed in Başar [1983]), the solution given in Lemma 7.1 cannot readily be used to obtain the solution to the soft constraint versions. In fact, it turns out that, for each of the soft constraint versions, the saddle-point property has to be established individually.

7.4.2. Solutions to the minimax problem under the three criteria

(a) Game G_{32}

Preliminary notation for Theorem 7.2.

We decompose the parameter space into three regions R1, R2 and R3:

$$R_{32}^1:\ \alpha \leqslant k_o$$

$$R_{32}^2:\ \alpha > k_o \text{ and } \alpha k_o - k_o^2 < \alpha^3 \sigma_w^2$$

and

$$R_{32}^3:\ \alpha > k_o \text{ and } \alpha k_o - k_o^2 \geqslant \alpha^3 \sigma_w^2 \quad .$$

We next introduce the scalar parameters P^*_o and λ^* by

$$P^*_o = \begin{cases} 0 & \text{in } R^1_{32} \\ (\mathrm{Max}\{0,\sigma_w((k_o^{-1}-\alpha^{-1})^{1/2}-\sigma_w)/((1-k_o/\alpha)^2)\})^{1/2} & \text{in } R^2_{32} \\ \alpha^{-1/2} & \text{in } R^3_{32} \end{cases}$$

and

$$\lambda^* = \begin{cases} -1 & \text{in } R^1_{32} \\ -k_o/\alpha & \text{in } R^2_{32} \text{ and } R^3_{32} \end{cases} .$$

Theorem 7.2. Game G_{32} admits two saddle-point solutions $(\gamma^*,\delta^*,\mu^*)$ and $(-\gamma^*,-\delta^*,\mu^*)$ where

(i) $\gamma^*(u) = P^*_o u$

(ii) μ^* is the Gaussian probability measure associated with the random variable

$$v = \beta^*(y) = \begin{cases} \lambda^* y & \text{in } R^1_{32} \text{ and } R^2_{32} \\ \lambda^* y + \eta^* & \text{in } R^3_{32} \end{cases}$$

where η^* is an independent Gaussian random variable, i.e.,

$$\eta^* \sim N(0,\sigma^2_{\eta^*}) = N(0,k_o/\alpha^2 - k_o^2/\alpha^3 - \sigma^2_w)$$

(iii) δ^* is the Bayes estimator for u under the least favorable distribution μ^*, computed as

$$\delta^*(z) = p^*z = \begin{cases} (P^*_o(1+\lambda^*)/(P^{*2}_o(1+\lambda^*)^2+\sigma_w^2))\ z & \text{in } R^1_{32} \text{ and } R^2_{32} \\ \\ (1/P^*_o)\ z & \text{in } R^3_{32}\ . \end{cases}$$

Proof. We shall establish separately the validity of the right-hand side (RHS) and left-hand side (LHS) inequalities of (7.18) in each of the Regions R^1_{32}, R^2_{32} and R^3_{32}.

Region R^1_{32}

(i) The RHS inequality:

With $\beta(y) = \beta^*(y) = -y$, the problem faced by the encoder-decoder pair is

$$\min_{\gamma,\delta} E\{(k_o-\alpha)\gamma^2(u)+1\}$$

which implies $\gamma^*(u) = 0$ and $\delta^*(z) = E[u \mid w] = 0$

(ii) The LHS inequality:

With $\gamma^* = \delta^* = 0$, the problem faced by the jammer is

$$\max_{\beta} E\{1 - \alpha\beta^2(y)\}$$

and since y=0 a.s., any choice of β with $E[\beta^2(y)]=0$ attains the maximum (which is 1), one such policy being $\beta(y) = -y$.

Region R^2_{32}

(i) The RHS inequality:

With the jammer's policy fixed as $\beta^*(y) = (-k_o/\alpha)y$, the problem faced by the encoder-decoder pair is

$$\min_{\gamma,\delta} E[(k_o-k_o^2/\alpha)\gamma^2(u) + (\delta(z)-u)^2]$$

with $z = (1-k_o/\alpha)\gamma(u) + w$.

Defining $\gamma'(u) = (1-k_o/\alpha)\gamma(u)$, the problem may be rewritten as

$$\min_{\gamma',\delta} E[k'(\gamma'(u))^2 + (\delta(z)-u)^2]$$

where $z = \gamma'(u)+w$. It follows from Lemma 7.2 that the optimal encoder-decoder policies are linear in the observations, and may be obtained by minimizing over $P_o^2 \geqslant 0$ the expression

$$J(P_o^2) = k_o(1-k_o/\alpha)P_o^2 + \sigma_w^2/(1-k_o/\alpha)^2 P_o^2 + \sigma_w^2 ,$$

which yields P^*_o as in the statement of the theorem.

(ii) The LHS inequality:

With the encoder-decoder policy fixed as given, the problem faced by the jammer is

$$\max_{\beta} E[(p^{*2}-\alpha)\beta^2(y) + 2p^*(p^*P^*_o-1)\beta(y)u] .$$

Under the assumption $p^{*2}-\alpha<0$, its solution is unique (because of strict concavity), and is given by

$$\beta(y) = [p^*(p^*P^*_o-1)/(P^*_o(\alpha-p^{*2}))]\, y = \lambda^* y .$$

With p^* as indicated, this condition becomes

$$-\lambda^*(1+\lambda^*) < \sigma_w^2/P^{*2}_o$$

which implies and is implied by

$$\alpha^3\sigma_w^2 > \alpha k_o - k_o^2 .$$

Region R_{32}^3

(i) The RHS inequality:

With the jammer's policy fixed as indicated, the problem faced by the encoder-decoder pair is

$$\min_{\gamma,\delta} E\{(k_o - k_o^2/\alpha)\gamma^2(u) + (\delta(z)-u)^2 - \alpha\eta^{*2}\} .$$

Using Lemma 7.2 as in Region R_{32}^2, we obtain the encoder policy to be linear, with the corresponding power level obtained by minimizing over P_o^2 the expression

$$J(P_o^2) = k_o(1-k_o/\alpha)P_o^2 + (\sigma_w^2 + \sigma_{\eta^*}^2)/[(1-k_o/\alpha)^2 P_o^2 + \sigma_w^2 + \sigma_{\eta^*}^2]$$

which requires (by differentiating with respect to P_o^2 and noting that the second-order condition for a minimum is always satisfied)

$$k_o(1-k_o/\alpha) - (\sigma_w^2+\sigma_{\eta^*}^2)(1-k_o/\alpha)^2/[(1-k_o/\alpha)^2 P_o^{*2}+\sigma_w^2+\sigma_{\eta^*}^2]^2 = 0$$

which is satisfied (uniquely) by $P_o^{*2} = 1/\alpha$. Furthermore,

$$\delta(z) = E[u \mid z] = \sqrt{\alpha}\, z .$$

(ii) The LHS inequality:

With the encoder-decoder policy fixed as indicated, the problem faced by the jammer is

$$\max_{\beta} E\{(p^{*2}-\alpha)\beta^2(y) + 2p^*(p^*P^*_o - 1)\beta(y)\, u\} .$$

Since $p^{*2} = \alpha$ and $p^* = 1/P^*_o$, this cost is independent of β, and hence the jammer's policy may be chosen as indicated.

(b) Game G_{33}

Consider the game of Type 3, under Criterion C3. The solution is provided in Theorem 7.3 below, after introducing some notation.

Preliminary Notation for Theorem 7.3.

We decompose the parameter space into Regions R_{33}^1 and R_{33}^2, as follows:

$$R_{33}^1:\ (P_o^2 \neq 1/\alpha) \cup (P_o^2 = 1/\alpha < 4\sigma_w^2)$$
$$R_{33}^2:\ P_o^2 = 1/\alpha \geqslant 4\sigma_w^2 .$$

We next introduce the scalar parameter λ^* by

$$\lambda^* = \begin{cases} \arg\max\limits_{\lambda} \{\sigma_w^2/[(1+\lambda)^2 P_o^2 + \sigma_w^2] - \alpha\lambda^2 P_o^2\} & \text{in } R_{33}^1 \\ \\ -\dfrac{1}{2} & \text{in } R_{33}^2 . \end{cases}$$

Note that $\lambda^* \in (-1,0)$.

Theorem 7.3. Game G_{33} admits two saddle-point solutions $(\gamma^*,\delta^*,\mu^*)$ and $(-\gamma^*,-\delta^*,\mu)$ where

(i) $\gamma^*(u) = P_o u$

(ii) μ^* is the Gaussian probability measure associated with the random variable

$$v = \beta^*(y) = \begin{cases} \lambda^* y & \text{in } R_{33}^1 \\ \\ \lambda^* y + \eta^* & \text{in } R_{33}^2 \end{cases}$$

where $\eta^* \sim N(0,\sigma_{\eta^*}^2) = N(0,(1/4\alpha - \sigma_w^2)^{1/2})$.

(iii) δ^* is the Bayes estimator for u under the least favorable distribution μ^* for v, computed as

$$\delta^*(z) = p^*z = \begin{cases} (P_o(1+\lambda^*)/[P_o^2(1+\lambda^*)^2+\sigma_w^2])\, z & \text{in } R_{33}^1 \\ \\ (1/P_o)\, z & \text{in } R_{33}^2 \,. \end{cases}$$

Proof. We shall establish separately the validity of the RHS and the LHS inequalities of (7.18) in Regions R_{33}^1 and R_{33}^2.

Region R_{33}^1

(i) The RHS inequality:

With the jammer's policy fixed as indicated, the problem faced by the encoder-decoder pair is

$$\min_{\gamma,\delta} E\{(\delta(z)-u)^2 - \alpha\lambda^{*2}\gamma^2(u)\}\,,\ E[\gamma^2(u)] \leqslant P_o^2\,,$$

where $z=(1+\lambda^*)\gamma(u)+w$. Noting that $\lambda^* \in (-1,0)$ it follows that

$$\min_{\gamma,\delta} E\{(\delta(z)-u)^2\}$$

under $E[\gamma^2(u)] \leqslant P_o^2$ is achieved by $\gamma(u)=P_o u$.

Now $\alpha\lambda^{*2}E\{\gamma^2(u)\}$ is also maximized when $\gamma(u)=P_o u$ and therefore we obtain the solution of the above problem as

$$\gamma(u) = P_o u$$

and

$$\delta(z) = E[u\,|\,z] = [P_o(1+\lambda^*)/(P_o^2(1+\lambda^*)^2+\sigma_w^2)]\ z\ .$$

(ii) The LHS inequality:

With the encoder-decoder policies fixed as indicated, the problem faced by the jammer is

$$\max_{\beta} E[(p^{*2}-\alpha)\beta^2(y) + 2p^*(p^*P_o-1)\beta(y)u] ,$$

the unique solution to which is

$$\beta^*(y) = [p^*(p^*P_o-1)/(P_o(\alpha-p^{*2}))]\, y = \lambda^* y ,$$

provided that

$$p^{*2} - \alpha < 0 ,$$

or equivalently

$$-\lambda^*(1+\lambda^*) < \sigma_w^2/P_o^2 . \tag{7.19}$$

We now show that the inequality (2) holds for all P_o^2 in R_{33}^1.

(a) First consider the case $P_o^2 = 1/\alpha < 4\sigma_w^2$. Note that under this condition the equation $\lambda^{*2} + \lambda^* + \sigma_w^2/P_o^2 = 0$ has no real roots, which implies that

$$\lambda^{*2} + \lambda^* + \sigma_w^2/P_o^2 > 0 ,$$

and hence $-\lambda^*(1+\lambda^*) < \sigma_w^2/P_o^2$.

(b) Now, consider the case $P_o^2 \neq 1/\alpha$. If $P_o^2 = 1/\alpha$ we know that λ satisfies

$$[(1+\lambda)^2P_o^2 + \sigma_w^2]^2 + (\sigma_w^2(1+\lambda)/\lambda)P_o^2 = 0 , \tag{7.20}$$

which after some manipulations yields

$$g(\lambda) = (\lambda(1+\lambda) + \sigma_w^2/P_o^2)(\sigma_w^2/P_o^2 + (1+\lambda)^3/\lambda) = 0 . \tag{7.21}$$

This function is illustrated in Figures 7.4a and 7.4b. The extrema are found at the intersections of the horizontal line at αA, the necessary condition being

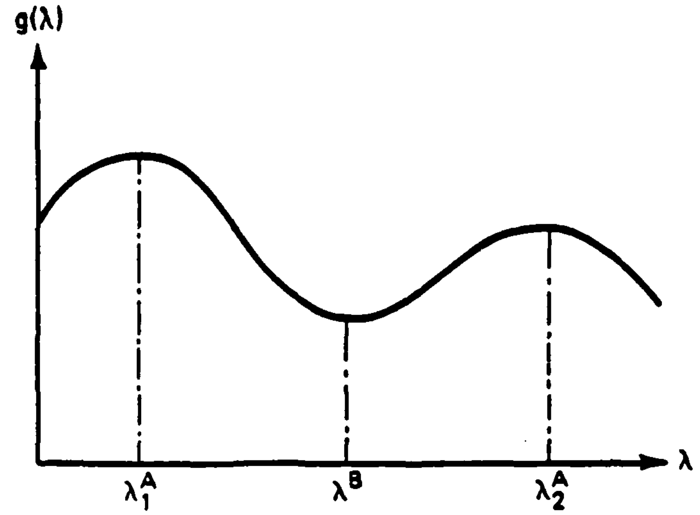

Figure 7.4a. The function $g(\lambda)$.

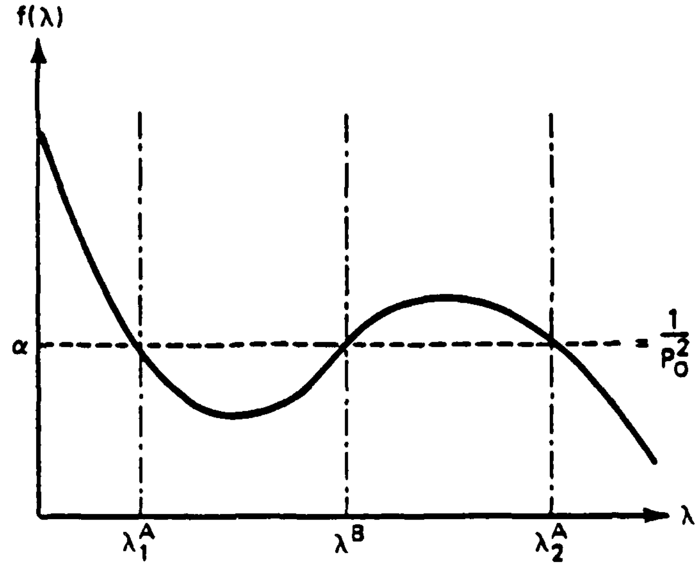

Figure 7.4b. The function $f(\lambda)$.

$$\alpha = -\sigma_w^2(1+\lambda)/[\lambda((1+\lambda)^2 P_o^2 + \sigma_w^2)]^2 \triangleq f(\lambda) .$$

The quadratic term of (7.21) corresponds to the maximizing roots, whereas the cubic term in (7.21) has only one real root which is the minimizing solution.

Let

$$A(\lambda) \triangleq \lambda(1+\lambda) + \sigma_w^2/P_o^2 \text{ (with roots } \lambda_1^A \text{ and } \lambda_2^A),$$

and

$$B(\lambda) \triangleq \sigma_w^2/P_o^2 + (1+\lambda)^3/\lambda \text{ (with real root } \lambda^B),$$

so that Equation (7.21) may be rewritten as

$$A(\lambda) \cdot B(\lambda) = 0 \quad .$$

From Figure 7.4 it is apparent that if $\alpha > 1/P_o^2$, we get a maximizing root smaller than λ_1^A, which satisfies (7.19), and if $\alpha < 1/P_o^2$, we get a maximizing root larger than λ_2^A, which satisfies (7.19), and thus the proof for Region R_{33}^1 is complete.

Region R_{33}^2

(*i*) The RHS inequality:

With the jammer's policy as indicated, the problem faced by the encoder-decoder pair is

$$\min_{\gamma,\delta} E\{(\delta(z)-u)^2 - \alpha\lambda^{*2}\gamma^2(u) - \alpha\sigma_{\eta^*}^2\} , E[\gamma^2(u)] \leqslant P_o^2$$

with $z = (1+\lambda^*)\gamma(u) + w + \eta^*$.

It follows then, as in the proof for Region R_{33}^1, that the required solution is

$$\gamma(u) = P_o u$$

and

$$\delta(z) = E[u \mid z] = (1/P_o) z .$$

(ii) The LHS inequality:

$$J(\gamma^*,\delta^*,\beta) \leqslant J(\gamma^*,\delta^*,\beta^*) \ .$$

With the encoder-decoder policies fixed as indicated we have $p^*P_o = 1$ and $p^{*2} = \alpha$. Therefore, the maximization problem faced by the jammer, i.e.,

$$\max_{\beta(y)} \ E[(p^{*2}-\alpha)\beta^2(y) + 2p^*(p^*P_o-1)\beta(y)u] \ ,$$

has a cost function which is independent of β which may, therefore, be chosen as in the statement of the theorem.

Remark 7.1. This solution for the jamming policy in Region R_{33}^2 is not unique. In fact, any pair $(\lambda^*,\sigma_{\eta^*})$ which satisfies

$$(1+\lambda^*)^2 - (1+\lambda^*) + \alpha(\sigma_w^2+\sigma_{\eta^*}^2) = 0$$

provides a solution, since this gives

$$p^* = (1+\lambda^*)P_o/((1+\lambda^*)^2P_o^2+\sigma_w^2+\sigma_{\eta^*}^2) = 1/P_o \ ,$$

thus yielding a cost which is independent of β. The specific choice of the solution pair in the statement of the theorem is motivated by continuity of λ^* at the boundary of R_{33}^1 and R_{33}^2.

(c) Game G_{34}

It turns out that Game G_{34} does not admit a saddle-point solution and hence the general approach used so far is not applicable. This is because the maximin value of the problem is not well defined. (In order to find the maximin value, we need to fix the jammer's policy and then minimize over the encoder-decoder policy; but the jamming policy itself depends on the encoder policy through the constraint, thus leading to ill-posedness.)

However, we can still obtain a minimax solution by proceeding in two stages. First, we formulate a related problem with an enlarged information structure for which we can show the existence of a saddle-point solution. We then show that the minimax value of the original problem and that of the problem with the enlarged information structure are identical, and hence the saddle-point solution found corresponds to the minimax solution desired.

Towards this end, suppose that in addition to having access to the random variable $y=\gamma(u)$, the jammer also has access to the policy γ, in other words, the jammer's policy $\tilde{\beta}$ belongs to the space of all Borel measurable mappings which are functions of both y and γ, i.e., $v=\tilde{\beta}(y,\gamma)$. The saddle-point solution to the new problem with this enlarged information structure is presented below in Proposition 7.1 after introducing some notation.

Preliminary notation for Proposition 7.1.

Let

$$P^*_o = \arg\min_{P_o \geq 0} f(P_o)$$

where

$$f(P_o) = \begin{cases} k_o P_o^2 + \sigma_w^2/[(P_o - P_J)^2 + \sigma_w^2] & \text{if } P_o \geq P_J \\ \\ k_o P_o^2 + 1 & \text{if } P_o < P_J \end{cases} .$$

Note that either $P^*_o = 0$ or $P^*_o > P_J$.

Furthermore, decompose the parameter space into Regions R^1_{34} and R^2_{34} characterized by

$$R_{34}^1 : P^*_o > P_J$$

$$R_{34}^2 : P^*_o = 0 \; .$$

Proposition 7.1. The problem with the enlarged information structure, just described, admits two saddle-point solutions $(\gamma^*, \delta^*, \mu^*)$ and $(-\gamma^*, -\delta^*, \mu^*)$ where

(i) $\gamma^*(u) = P^*_o u$

(ii) μ^* is the Gaussian probability measure associated with the random variable

$$v = \tilde{\beta}^*(y,\gamma) = \begin{cases} -(P_J/\|y\|)\, y & \text{in } R_{34}^1 \\ \\ -y & \text{in R2} \end{cases}$$

(iii) $\delta^*(z) = p^* z$

$$= \begin{cases} [(P^*_o - P_J)/((P^*_o - P_J)^2 + \sigma_w^2)]\, z & \text{in } R_{34}^1 \\ \\ 0 & \text{in } R_{34}^2 \end{cases} .$$

Proof. We shall establish separately the validity of the RHS and the LHS inequalities of (7.18) in R_{34}^1 and R_{34}^2. (The key observation here is that the jammer must know his opponent's policy in order to determine a minimax solution. The enlarged information structure enables the jammer to use the policies as in (ii) above.)

Region R_{34}^1.

(i) The RHS inequality:

The problem faced by the encoder-decoder pair, with the jammer's policy fixed as indicated, is

$$\min_{\gamma,\delta} E[k_o\gamma^2(u) + (\delta(z)-u)^2]$$

where

$$z = (1 - P_J/\|y\|)y + w \ .$$

Let us first consider the Problem P1:

$$\min_{\gamma,\delta} E\{(\delta(z) - u)^2\} \ ; \ E[\gamma^2(u)] \leqslant P_o^2 \ .$$

We assert that the minimum value of this problem is achieved by some γ satisfying $E[\gamma^2(u)] = P_o^2$. To verify this, we first assume the contrary, that the minimum is achieved by a γ satisfying $E[\gamma^2(u)] = P^2$, where $0 < P_J < P < P_o$. The solution to the Problem P2:

$$\min_{\gamma,\delta} E\{(\delta(z) - u)^2\} \ ; \ E[\gamma^2(u)] = P^2 \ ,$$

is then the same as that for the Problem P1. However, for Problem P2 we know that

$$z = (1-P_J/P)\gamma(u) + w \ .$$

By defining $\gamma'(u) = (1-P_J/P)\gamma(u)$, and using Lemma 7.3, we find that the solution to Problem P2 is linear and yields an optimal performance of

$$\sigma_w^2/((P-P_J)^2+\sigma_w^2) \ .$$

By using a linear solution $y = P_o\, u$ for Problem P1, we obtain the performance

$$\sigma_w^2/((P_o-P_J)^2+\sigma_w^2)$$

which is clearly lower than the one obtained for Problem P2, implying that the minimum cannot be achieved for $E[\gamma^2(u)] = P^2 < P_o^2$. We now have

$$\min_{\substack{\gamma,\delta \\ (E[\gamma^2(u)] = P_o^2 < P_J^2)}} E[k_o\gamma^2(u) + (\delta(z) - u)^2]$$

$$\geqslant k_o P_o^2 + \sigma_w^2/[(P_o - P_J)^2 + \sigma_w^2]$$

$$\geqslant \min_{P_o > P_J} (k_o P_o^2 + \sigma_w^2/[(P_o - P_J)^2 + \sigma_w^2)]$$

$$= k_o P{*}_o^2 + \sigma_w^2/[(P{*}_o - P_J)^2 + \sigma_w^2]$$

where $P{*}_o > P_J$ necessarily exists in R_{34}^1. With $\gamma(u) = P{*}_o\, u$ and $\delta(z) = E[u \mid z]$ the above inequalities are all tight, implying that the minimum is indeed achieved by the policies given in the statement of the theorem.

(ii) The LHS inequality:

With γ^* and δ^* fixed as indicated, the problem faced by the jammer is

$$\max_{\beta} \{p^{*2}\beta^2(y) + 2p^*(p^*P{*}_o - 1)\beta(y)\, u\} ,$$

subject to the constraint $E[\beta^2(y)] \leqslant P_J^2$. Clearly the optimal policy for the jammer is to use a function, linear in u, at the maximum permissible power level, which, under the enlarged information structure, is

$$\tilde{\beta}^*(y,\gamma) = -(P_J/\|y\|)\, y .$$

Region R_{34}^2

(i) The RHS inequality:

With $\beta^*(y) = -y$, the problem faced by the encoder-decoder pair is

$$\min_{\gamma,\delta} E[k_o\gamma^2(u) + 1] ,$$

the solution to which clearly requires $\gamma(u) = 0$ and

$$\delta(z) = E[u \mid z] = 0 \quad .$$

(ii) The LHS inequality:

With the encoder-decoder policies fixed at zero, the cost becomes independent of the jammer's policy, who may now use $\beta(y) = -y$ as the policy which yields the saddle-point solution.

We finally argue that the minimax values of the two problems with and without the enlarged information structure are the same. With γ fixed, let β belong to the space of Borel measurable functions of y satisfying $E[\beta^2(y)] \leqslant P_J^2$, and let $\tilde{\beta}$ be in the space of Borel measurable functions of y and γ satisfying $E[\tilde{\beta}^2(y,\gamma)] \leqslant P_J^2$. Then

$$\min_{\gamma,\delta} \max_{\tilde{\beta}} J(\gamma,\delta,\tilde{\beta}) = \min_{\gamma,\delta} \max_{\beta} J(\gamma,\delta,\beta) \ ,$$

since for fixed γ and δ the two *inner maximization* problems are identical.

We therefore have Theorem 7.4 below.

Theorem 7.4. Game G_{34} admits a minimax solution which is equivalent to the saddle-point solution given in Proposition 7.1.

□

7.5. Minimax and Maximin Strategies for Games G_{22}, G_{23} and G_{24}

7.5.1. Minimax strategies

If the encoding policy γ is restricted to be deterministic, then to every jamming policy β that is a function of the encoder output $\gamma(u)$, there corresponds a jamming policy $\tilde{\beta} = \beta \cdot \gamma$ which represents the same random variable

$$v = \tilde{\beta}(u) = \beta(\gamma(u)) \ . \tag{7.22}$$

Since γ is not necessarily invertible, we have the inequality

$$\sup_{\beta} J(\gamma,\delta,\beta(\gamma(u)) \leqslant \sup_{\tilde{\beta}} J(\gamma,\delta,\tilde{\beta}(u)) \tag{7.23}$$

for every pair (γ,δ), where γ is restricted to be deterministic. Taking the infimum of both sides over (γ,δ) we obtain for k=2,3,4,

$$\bar{J}_{3k} = \inf_{(\gamma,\delta)} \sup_{\beta} J_k(\gamma,\delta,\beta(\gamma(u)) \leqslant \bar{J}_{2k} \tag{7.24}$$

where $\bar{J}_{3k}$ is the upper value of a similar game with fidelity criterion C_k, with the difference that in this new game the jammer has access to the output of the encoder rather than the input. Thus the upper value of a game of Type 2 is bounded from below by the upper value of the corresponding game of Type 3. We shall next show that this inequality is in fact an equality, and provide for each $k \in \{2,3,4\}$, a set of minimax strategies that achieve this value.

Preliminary notation for Theorem 7.5

(a) Game G_{22}

Define Regions $\bar{R}^1_{22}$, $\bar{R}^2_{22}$ and $\bar{R}^3_{22}$ by

$$\begin{aligned} &\bar{R}^1_{22} : \alpha \leqslant k_0 \\ &\bar{R}^2_{22} : \alpha > k_0 \text{ and } \alpha k_0 - k_0^2 < \alpha^3 \sigma_w^2 \\ &\bar{R}^3_{22} : \alpha > k_0 \text{ and } \alpha k_0 - k_0^2 \geqslant \alpha^3 \sigma_w^2 . \end{aligned} \tag{7.25}$$

Now define P^*_0 and λ^* by

$$P^*_0 = \begin{cases} 0 & \text{in } \overline{R}^1_{22} \\ (\mathrm{Max}\{0, \sigma_w((k_0^{-1}-\alpha^{-1})^{1/2}-\sigma_w)/((1-k_0/\alpha)^2)\})^{1/2} & \text{in } \overline{R}^2_{22} \\ \alpha^{-1/2} & \text{in } \overline{R}^3_{22} \end{cases} \tag{7.26}$$

$$\lambda^* = \begin{cases} -1 & \text{in } \overline{R}^1_{22} \\ -k_0/\alpha & \text{in } \overline{R}^3_{22} \text{ and } \overline{R}^3_{22} \end{cases} . \tag{7.27}$$

Let $(\overline{\gamma}^*, \overline{\delta}^*, \overline{\beta}^*)$ be defined by

$$\overline{\gamma}^*(u) = P^*_0 u \qquad \text{in } \overline{R}^1_{22}, \overline{R}^2_{22} \text{ and } \overline{R}^3_{22} \tag{7.28}$$

$$\overline{\delta}^*(z) = p^* z = \begin{cases} (P^*_0(1+\lambda^*)/(P^{*2}_0(1+\lambda^*)^2+\sigma^2_w))z & \text{in } R^1_{22} \text{ and } \overline{R}^2_{22} \\ (1/P^*_0)z & \text{in } \overline{R}^3_{22} \end{cases} \tag{7.29}$$

$$\overline{\beta}^*(u) = \begin{cases} 0 & \text{in } \overline{R}^1_{22} \\ -(k_0/\alpha)P^*_0 u & \text{in } \overline{R}^2_{22} \\ -(\dfrac{k_0}{\alpha})P^*_0 u + \eta^* & \text{in } \overline{R}^3_{22} \end{cases} \tag{7.30}$$

where η^* is an independent Gaussian random variable, $\eta^* \sim N(0, k_0/\alpha^2 - k_0^2/\alpha^3 - \sigma^2_w)$.

(b) Game G_{23}

Define Regions $\overline{R}^1_{23}$ and $\overline{R}^2_{23}$ by

$$\bar{R}^1_{23} : (P_0^2 \neq \frac{1}{\alpha}) \cup (P_0^2 = \frac{1}{\alpha} < 4\sigma_w^2)$$
$$\bar{R}^2_{23} : P_0^2 = \frac{1}{\alpha} \geqslant 4\sigma_w^2 . \tag{7.31}$$

Now define λ^* by

$$\lambda^* = \begin{cases} \arg\max\limits_{\lambda} (\sigma_w^2/((1+\lambda)^2 P_0^2 + \sigma_w^2) - \alpha\lambda^2 P_0^2) & \text{in } \bar{R}^1_{23} \\ -\frac{1}{2} & \text{in } \bar{R}^1_{23} \end{cases} . \tag{7.32}$$

Let $(\bar{\gamma}^*, \bar{\delta}^*, \bar{\beta}^*)$ be defined by

$$\bar{\gamma}^*(u) = P_0 u \qquad \text{in } \bar{R}^1_{23} \text{ and } \bar{R}^2_{23} \tag{7.33}$$

$$\bar{\delta}^* = \begin{cases} (P_0(1+\lambda^*)/(P_0^2(1+\lambda^*)^2 + \sigma_w^2))z & \text{in } R^1_{23} \\ (1/P_0)z & \text{in } \bar{R}^2_{23} \end{cases} \tag{7.34}$$

$$\bar{\beta}^* = \begin{cases} \lambda^* P_0 u & \text{in } R^1_{23} \\ \lambda^* P_0 u + \eta^* & \text{in } \bar{R}^2_{23} \end{cases} \tag{7.35}$$

where η^* is an independent Gaussian random variable, $\eta^* \sim N(0, 1/4\alpha - \sigma_w^2)^{1/2})$.

(c) Game G_{24}

Let

$$P^*_0 \equiv \arg\min_{P_0 \geqslant 0} f(P_0) \tag{7.36}$$

where

$$f(P_0) \equiv \begin{cases} k_0P_0^2+\sigma_w^2/((P_0-P_J)^2+\sigma_w^2) & \text{if } P_0 \geqslant P_J \\ k_0P_0^2 & \text{if } P_0 < P_J \end{cases} . \tag{7.37}$$

Note that either $P^*_0 > P_J$ or $P^*_0 = 0$.

Define Regions $\bar{R}_{24}^1$ and $\bar{R}_{24}^2$ by

$$\begin{aligned} &\bar{R}_{24}^1 : P^*_0 > P_J \\ &\bar{R}_{24}^2 : P^*_0 = 0 . \end{aligned} \tag{7.38}$$

Let $(\bar{\gamma}^*, \bar{\delta}^*, \bar{\beta}^*)$ be defined by

$$\bar{\gamma}^*(u) = P^*_0 u \tag{7.39}$$

$$\bar{\delta}^*(u) = \begin{cases} ((P^*_0-P_J)/((P^*_0-P_J)^2+\sigma_w^2))z & \text{in } R_{24}^1 \\ 0 & \text{in } \bar{R}_{24}^2 \end{cases} \tag{7.40}$$

$$\bar{\beta}^*(u) = \begin{cases} -P_J u & \text{in } R_{24}^1 \\ 0 & \text{in } \bar{R}_{24}^2 \end{cases} . \tag{7.41}$$

Theorem 7.5. Sets of minimax solutions for games G_{22}, G_{23} and G_{24} are provided by $(\bar{\gamma}^*, \bar{\delta}^*, \bar{\beta}^*)$ and $(-\bar{\gamma}^*, -\bar{\delta}^*, -\bar{\beta}^*)$ with $\bar{\gamma}^*, \bar{\delta}^*$ and $\bar{\beta}^*$ defined above in the various regions under each of the three fidelity criteria.

Proof. We shall first establish that

$$\bar{J}_{2k} \leqslant \sup_{\beta} J_k(\bar{\gamma}^*, \bar{\delta}^*, \beta(u)) = J_k(\bar{\gamma}^*, \bar{\delta}^*, \bar{\beta}^*) = \bar{J}_{3k} . \tag{7.42}$$

The first inequality in (7.42) is immediate because

$$\bar{J}_{2k} \leqslant \sup_{\beta} J_k(\gamma,\delta,\beta(u))$$

for all pairs (γ,δ). $\bar{J}_{3K}$ is the minimax value for the corresponding game of Type 3, and we shall prove the equality in (7.42) by considering individually the various regions under each of the criteria and comparing with the minimax strategies for problems of Type 3 available from Section 7.4.

(a) Game G_{22}

(i) Region $\bar{R}^1_{22}$

$$J_2(\bar{\gamma}^*,\bar{\delta}^*,\beta) = E[1-\alpha\beta^2(u)] .$$

Therefore, $\bar{\beta}^*(u) = 0$ and

$$\sup_{\beta} J_2(\bar{\gamma}^*,\bar{\delta}^*,\beta(u)) = \bar{J}_{32} \quad \text{in } \bar{R}^1_{22} .$$

(ii) *Region* $\bar{R}^2_{22}$

The problem faced by the jammer is

$$\max_{\beta} E[(p^{*2}-\alpha)\beta^2(u)+2p^*(p^*P^*_0-1)\beta(u)u]$$

for which the solution is

$$\bar{\beta}^*(u) = -(k_0/\alpha)P^*_0 u .$$

Noting that this provides the same value as the minimax value for Game G_{32} in Region $\bar{R}^2_{22}$, we have

$$\sup_{\beta} J_2(\bar{\gamma}^*,\bar{\delta}^*,\beta(u)) = \bar{J}_{32} \quad \text{in } \bar{R}^2_{22} .$$

(iii) *Region* $\overline{R}_{22}^{3}$

The problem faced by the jammer is

$$\max_{\beta} E[(p^{*2}-\alpha)\beta^2(u)+2p^*(p^*P^*_0-1)\beta(u)u] ,$$

and the cost is independent of β, the value being the same as the minimax value for Game G_{32} in $\overline{R}_{22}^{3}$, i.e.,

$$\sup_{\beta} J_2(\overline{\gamma}^*,\overline{\delta}^*,\beta(u)) = \overline{J}_{32} \quad \text{in } \overline{R}_{22}^{3} .$$

We remark that while β may be chosen arbitrarily to satisfy the equality, the specific choice of $\overline{\beta}^*$ as indicated provides the saddle-point solution for Game G_{23} and further provides continuity of the jamming policy at the boundary of Regions $\overline{R}_{22}^{2}$ and $\overline{R}_{22}^{3}$.

(*b*) *Game* G_{23}

(i) *Region* $\overline{R}_{23}^{1}$

The problem faced by the jammer is

$$\max_{\beta} E[(p^{*2}-\alpha)\beta^2(u)+2p^*(p^*P_0-1)\beta(u)u] ,$$

the solution to which is

$$\overline{\beta}^*(u) = \frac{p^*(p^*P_0-1)}{(\alpha-p^{*2})} u = \lambda^* P_0 u .$$

Since this solution provides the same value as the minimax strategy for G_{33} we have

$$\sup_{\beta} J_3(\overline{\gamma}^*,\overline{\delta}^*,\beta(u)) = \overline{J}_{33} \quad \text{in } \overline{R}_{23}^{1} .$$

(ii) *Region* $\bar{R}_{23}^{2}$

The problem faced by the jammer is to minimize a cost function which is independent of β as in the case of Region $\bar{R}_{22}^{3}$ and we have

$$\sup_{\beta} J_3(\bar{\gamma}^*,\bar{\delta}^*,\beta(u)) = \bar{J}_{33} \text{ in } \bar{R}_{23}^{2} .$$

Again β may be chosen arbitrarily; the specific choice here is due to the same reasons as remarked for Region $\bar{R}_{22}^{3}$.

(c) *Game* G_{24}

(i) *Region* $\bar{R}_{24}^{1}$

The problem faced by the jammer is

$$\max_{\beta} E\{p^{*2}\beta^2(u)+2p^*(p^*P^*_0-1)\beta(u)u\} ,$$

subject to $E[\beta^2(u)] \leqslant P_J^2$.

Clearly, the optimal policy for the jammer is to use a linear function of u at the maximum permissible power level, which gives

$$\bar{\beta}^*(u) = -P_J u$$

and provides the same value as the minimax strategy for Game G_{34}, implying that

$$\sup_{\beta} J_4(\bar{\gamma}^*,\bar{\delta}^*,\beta(u)) = \bar{J}_{34} \text{ in } \bar{R}_{24}^{1} .$$

(ii) *Region* $\bar{R}_{24}^{2}$

With the encoding-decoding policy pair fixed at zero, the cost is independent of the jammer's policy which may be chosen arbitrarily to provide

$$\sup_{\beta} J_4(\bar{\gamma}^*,\bar{\delta}^*,\beta(u)) = \bar{J}_{34} \text{ in } \bar{R}^2_{34} .$$

We now see that in all cases we have

$$\sup_{\beta} J_k(\bar{\gamma}^*,\bar{\delta}^*,\beta(u)) = \bar{J}_{3k} ,$$

and thus (7.42) holds.

Using (7.42) along with (7.24) it follows that $\bar{J}_{2k} = \bar{J}_{3k}$ for $k \in \{2,3,4\}$ and that the strategy sets given are sets of minimax policies.

□

7.5.2. Maximin Strategies

Since $\Gamma_e \supset \Gamma_{ed}$ we have

$$\inf_{\gamma\in\Gamma_e,\delta\in\Gamma_r} J_k(\gamma,\delta,\beta) \leqslant \inf_{\gamma\in\Gamma_{ed},\delta\in\Gamma_r} J_k(\gamma,\delta,\beta) \tag{7.43}$$

for all $\beta \in \Gamma_j$. Taking the supremum of both sides over $\beta \in \Gamma_j$, we get

$$\underline{J}_{1k} \leqslant \underline{J}_{2k} , \tag{7.44}$$

i.e., the maximin value for a game of Type 2 is bounded from below by the maximin value of the corresponding game of Type 1 (which is also the saddle-point value of the game of Type 1).

Preliminary notation for Theorem 7.6

(a) Game G_{22}

Define Regions R^1_{12}, R^2_{12} and R^3_{12} as in (7.10), and define (P^*_0,P^*_j) by (7.11).

(b) Game G_{23}

Define Regions R^1_{13} and R^2_{13} as in (7.12) and define (P^*_0, P^*_J) by (7.13).

(c) Game G_{24}

Define Regions R^1_{14} and R^2_{14} as in (7.14) and define (P^*_0, P^*_J) by (7.15).

Theorem 7.6. Sets of maximin solutions for games G_{22}, G_{23} and G_{24} are provided by $(\underline{\gamma}^*, \underline{\delta}^*, \underline{\beta}^*)$ and $(-\underline{\gamma}^*, -\underline{\delta}^*, \underline{\beta}^*)$ where

$$\underline{\gamma}^*(u) = P^*_0 u \tag{7.45a}$$

$$\underline{\delta}^*(z) = (P^*_0/(P^{*2}_0 + P^{*2}_J + \sigma^2_w))z \tag{7.45b}$$

and

$$\beta^*(u) = \eta^* \tag{7.45c}$$

where η^* is a zero mean Gaussian random variable with variance P^{*2}_J which is independent of u and w.

Proof. Note that

$$\underline{J}_{2k} = \sup_{\beta \in \Gamma^k_j} \inf_{\substack{\gamma \in \Gamma^k_{ed} \\ \delta \in \Gamma^k_r}} J_k(\gamma, \delta, \beta) \leqslant \sup_{\beta \in \Gamma^k_j} \inf_{\substack{\gamma \in \Gamma^{kl}_{ed} \\ \delta \in \Gamma^{kl}_r}} J_k(\gamma, \delta, \beta) \tag{7.46}$$

where Γ^{kl}_{ed} and Γ^{kl}_r are the sets of all permissible *linear* policies for the encoder and the decoder, respectively, and the inequality follows because the infimum on the right side is taken over a smaller set. We next show that the expression on the right side of the inequality in (7.46) is $\underline{J}_{1k}$, i.e., the maximin (or saddle-point) value for the corresponding game of Type 1.

(a) Game G_{22}

Let $\gamma(u) = \epsilon u$ and $\delta(z) = \Delta z$. Then

$$J_2(\gamma,\delta,\beta) = (1-\Delta\epsilon)^2 + \Delta^2\sigma_w^2 + \Delta^2 E[\beta^2(u)]$$
$$+ 2\Delta(\epsilon\Delta-1)E[u\beta(u)] - \alpha E[\beta^2(u)] + k_0\epsilon^2 .$$

Consider the minimization of the above functional over (ϵ,Δ). Using $\frac{\partial J_2}{\partial \Delta}|_{\Delta *} = 0$ and noting that $\frac{\partial^2 J_2}{\partial \Delta^2} > 0$, we obtain

$$\Delta * = \frac{\epsilon + E[u\beta(u)]}{\epsilon^2 + E[\beta^2(u)] + \sigma_w^2 + 2\epsilon E[u\beta(u)]}$$

as the unique minimizing $\Delta *$ for every fixed ϵ. Therefore,

$$J_2|_{\Delta *} = K_0\epsilon^2 + \frac{E[\beta^2(u)] + \sigma_w^2 - (E[u\beta(u)])^2}{\epsilon^2 + E[\beta^2(u)] + \sigma_w^2 + 2\epsilon E[u\beta(u)]} - \alpha E[\beta^2(u)] .$$

We next minimize $J_2|_{\Delta *}$ with respect to ϵ. In particular, if $E[u\beta(u)] = 0$ we get

$$\epsilon *^2 = \mathrm{Max}\left\{0 , \left(\frac{E[\beta^2(u)] + \sigma_w^2}{k_0}\right)^{1/2} - (E[\beta^2(u)] + \sigma_w^2)\right\}$$

and, in general, we have

$$J|_{\Delta *,\epsilon *} = k_0\epsilon *^2 + \frac{E[\beta^2(u)] + \sigma_w^2 - (E[u\beta(u)])^2}{\epsilon *^2 + E[\beta^2(u)] + \sigma_w^2 + 2\epsilon * E[u\beta(u)]} - \alpha E[\beta^2(u)]$$

and we note that $\mathrm{sgn}(\epsilon *) = -\mathrm{sgn}E[u\beta(u)]$.

To maximize $J|_{\Delta *,\epsilon *}$ as a function of $E[u\beta(u)]$ we differentiate with respect to $E[u\beta(u)]$ and find that

$$\frac{dJ|_{\Delta*,\epsilon*}}{d(E[u\beta(u)])} > 0 \qquad \text{if } E[u\beta(u)] < 0$$

and

$$\frac{dJ|_{\Delta*,\epsilon*}}{d(E[u\beta(u)])} < 0 \qquad \text{if } E[u\beta(u)] > 0 .$$

Therefore, $J|_{\Delta*,\alpha*}$, which is continuous in $E[u\beta(u)]$, admits its maximum at $E[u\beta(u)]=0$ and we next maximize over $E[\beta^2(u)]$ to find that

$$E[\beta^2(u)] = \text{Max}\{0\,,\ \frac{\epsilon*}{\alpha^{1/2}} - (\epsilon*^2+\sigma_w^2)\}\ .$$

Note that if $\epsilon*=0$, then $E[\beta^2(u)]$ is necessarily zero, which is the case if $\frac{1}{k_0} \leqslant \sigma_w^2$, i.e., Region R_{12}^1.

Next note that $\epsilon*$ could be nonzero with $E[\beta^2(u)]=0$, which occurs if

$$\epsilon*^2 = \frac{\sigma_w}{k_0^{1/2}} - \sigma_w^2$$

along with

$$\frac{\epsilon*^2}{\alpha} < (\epsilon* + \sigma_w^2)^2$$

which together yield

$$\frac{k_0}{(k_0+\alpha)^2} \leqslant \sigma_w^2 < \frac{1}{k_0}\ ,$$

i.e., we have Region R_{12}^2.

Finally note that in Region R_{12}^3 where

$$\sigma_w^2(k_0+\alpha)^2 < k_0 ,$$

both $\epsilon*$ and $E[\beta^2(u)]$ are nonzero.

Observing that the jammer's maximization problem is solved by using any second-order random variable with the appropriate variance, it follows that a Gaussian random variable as indicated is an appropriate choice. Thus, with the encoding-decoding policy restricted to be linear, the cost obtained by using the jammer's maximinimizing policy is the same as the maximin value of Game G_{12},

(b) Game G_{23}

With $\gamma(u)=\epsilon u$ and $\delta(z)=\Delta z$

$$J_3(\gamma,\delta,\beta)=(1-\Delta\epsilon)^2+\Delta^2\sigma_w^2+\Delta^2E[\beta^2(u)] \\ +2\Delta(\epsilon\Delta-1)E[u\beta(u)]-\alpha E[\beta^2(u)]$$

and we obtain as before

$$J_3|_{\Delta*}=\frac{E[\beta^2(u)]+\sigma_w^2-(E[u\beta(u)])^2}{\epsilon^2+E[\beta^2(u)]+\sigma_w^2+2\epsilon E[u\beta(u)]}-\alpha E[\beta^2(u)]$$

and minimization of $J|_{\Delta*}$ with respect to ϵ yields

$$\epsilon*=\begin{cases} P_0 & \text{if } E[u\beta(u)]>0 \\ -P_0 & \text{if } E[u\beta(u)]<0\,. \\ P_0 \text{ or } -P_0 & \text{if } E[u\beta(u)]=0 \end{cases}$$

(Recall that we require $|\epsilon^2| \leqslant P_0^2$). Now,

$$J_3|_{\Delta^*,\epsilon^*} = \begin{cases} \dfrac{E[\beta^2(u)]+\sigma_w^2-(E[u\beta(u)])^2}{P_0^2+E[\beta^2(u)]+\sigma_w^2-2P_0E[u\beta(u)]} - \alpha E[\beta^2(u)] & \text{if } E[u\beta(u)] \leqslant 0 \\ \\ \dfrac{E[\beta^2(u)]+\sigma_w^2-(E[u\beta(u)])^2}{P_0^2+E[\beta^2(u)]+\sigma_w^2+2P_0E[u\beta(u)]} - \alpha E[\beta^2(u)] & \text{if } E[u\beta(u)] \geqslant 0 \end{cases} .$$

Noting that the derivative of this expression with respect to $E[u\beta(u)]$ is increasing for negative $E[u\beta(u)]$ and decreasing for positive $E[u\beta(u)]$, we find that the unique maximum is at $E[u\beta(u)] = 0$.

To further maximize with respect to $E[\beta^2(u)]$ we take the derivative and get

$$E[\beta^2(u)] = \frac{P_0}{\alpha^{1/2}} - (P_0^2 + \sigma_w^2) \quad \text{if } \left(\frac{P_0}{\alpha^{1/2}}\right) > (P_0^2 + \sigma_w^2)$$

(i.e., Region R_{13}^2), and zero otherwise (i.e., Region R_{13}^1).

Also note that with the encoding-decoding policy restricted to be linear, the cost obtained by using the jammer's maximizing policy is the same as the maximin value for Game G_{13}.

(c) Game G_{24}

With $\gamma(u) = \epsilon u$ and $\delta(z) = \Delta z$

$$J_4|_{\Delta^*} = k_0 + \frac{E[\beta^2(u)]+\sigma_w^2-(E[u\beta(u)])^2}{\epsilon^2+E[\beta^2(u)]+\sigma_w^2+2\epsilon E[u\beta(u)]}$$

and

$$J_4\big|_{\Delta^*,\epsilon^*} = k_0 \epsilon^{*2} + \frac{E[\beta^2(u)] + \sigma_w^2 - (E[u\beta(u)])^2}{\epsilon^{*2} + E[\beta^2(u)] + \sigma_w^2 + 2\epsilon^* E[u\beta(u)]}$$

with $\mathrm{sgn}(\epsilon^*) = -\mathrm{sgn}\, E[u\beta(u)]$.

We differentiate $J_4\big|_{\Delta^*,\epsilon^*}$ with respect to $E[u\beta(u)]$ to find that the maximizing value is $E[u\beta(u)] = 0$. Further since $J_4\big|_{\Delta^*,\epsilon^*}$ is an increasing function of $E[u\beta(u)]$, its maximum over $\beta(u)$ subject to $E[u\beta(u)] \leqslant P_J^2$ is attained at $E[\beta^2(u)] = P_J^2$. Any second-order random variable with the appropriate variance may be used as the maximinimizing solution, yielding a cost which is the same as the maximin value for Game G_{14}.

We have thus shown that for each of the games the expression on the right side of the equality in (7.46) is $\underline{J}_{1K}$, i.e., we have

$$\underline{J}_{2K} \leqslant \underline{J}_{1K}\,. \tag{7.47}$$

Now using (7.47), along with (7.45), we find that $\underline{J}_{2K} = \underline{J}_{1K}$, and the strategies indicated are indeed the maximin strategies.

□

7.6. Conclusion

In this chapter general classes of communication games have been studied, which include situations where the encoder mapping may be random or restricted to be deterministic, the jammer may tap the input to the encoder or the input to the channel, and the power constraints on the encoder and the jammer may be hard or soft. For each of the above cases, minimax, maximin or saddle-point solutions have been provided.

The important results obtained are

(i) Games of Type 1 where the encoder structure is allowed to be probabilistic and the jammer taps the input to the encoder, admit saddle-point solutions under all fidelity criteria considered.

(ii) Games of Type 2 where the encoder structure is restricted to be deterministic do not admit a saddle-point solution; however, both minimax and maximin strategies may be found in this case.

(iii) The minimax value for a game of Type 2 coincides with the minimax value of the corresponding game of Type 3, and the maximin value for a game of Type 2 coincides with the maximin (saddle-point) value of the corresponding game of Type 1.

The least favorable jamming noise for games of Type 1 is Gaussian, and this adds to the list of results previously available in the literature where the Gaussian distribution has been shown to be extremal. However, it is important to note that for games of Type 1 the least favorable jamming noise is independent of the input, whereas for games of Type 3 this least favorable noise is correlated with the output of the encoder.

It is noteworthy that the analysis here cannot be trivially extended when the input sequence is vector valued or the number of channels is more than one, since the counterpart of the standard Gaussian test channel does not admit a simple linear coding scheme in the vector case.

CHAPTER 8

RECAPITULATION AND CONCLUSIONS

In this thesis we considered the problem of simultaneously designing communication and control strategies for decentralized systems. The main thrust was towards identifying classes of problems with linear dynamics, quadratic loss functionals and Gaussian statistics for which the optimality of linear strategies could be established. The general approach used consisted of first finding a lower bound on the cost, and then constructing strategies which achieved this lower bound. For some instances of the cases in which linear strategies failed to provide globally optimal solutions, explicit nonlinear strategies were obtained to demonstrate the inferiority of linear designs.

In Chapter 1 we introduced scenarios in which the simultaneous design of communication strategies and control policies may be desired, and provided a discussion of various types of information structures associated with stochastic team problems. Some issues of computational complexity, which arise when the numerical derivation of the optimal team solution is attempted, were also discussed in Chapter 1.

In Chapter 2 we formulated and analyzed some fundamental classes of stochastic team problems with two decentralized agents. We identified those instances of the general problem for which the optimal solutions are linear. It was shown that if the first agent observes an uncorrupted version of the variable to be transmitted, or if the channel noise is uncorrelated with the input, then the decentralized team problem admits an optimal solution which is linear in the observation variables, and the linear coefficients may be found by solving for the roots of the fifth-order polynomial. For some instances where the first agent observes a noise-corrupted version of the variable to be transmitted, with

the channel noise also correlated with this variable, we provided nonlinear strategies that outperform the optimal linear strategies. We also commented on some aspects of the difficulties associated with Witsenhausen's problem, which is one of the most important and most referenced counterexamples in stochastic control, refuting the common belief (prior to 1968) that all linear quadratic Gaussian control problems admit linear solutions.

In Chapter 3 we considered stochastic dynamic team problems where at each step two consecutive decisions must be taken, one being what information-bearing signal to transmit, and the other regarding what control action to exert. Such problems arise in the simultaneous optimization of both the observation and the control sequences in stochastic systems. We solved the problem completely for first-order systems under quadratic cost criteria. This was done by first constructing an equivalent problem having a cost function consisting of a sum of squared differences, and then solving this equivalent problem by using some bounds from Information Theory. For cases with hard power constraints, it was shown that the optimum measurement strategy is to linearly amplify the innovation at each stage to the maximum permissible power level. For cases with soft power constraints, the structure of the solution was found to be similar, with the optimum power levels being found via solving a nonlinear optimal control problem, this in turn being done by using a dynamic program. The results were then extended to cases with an infinite time horizon and a discounted cost functional, and the existence of optimal stationary policies for these problems was established.

In Chapter 4 we considered stochastic dynamic decision problems requiring simultaneous optimization of both the observation and the control sequences for second- and higher-order systems under quadratic cost criteria. We showed that for some of the simplest of such problems, involving a second-order system, the optimal linear strategies may

be outperformed by appropriately chosen nonlinear strategies. We considered optimality over the affine class for problems involving a general j-th order model, and showed that within this class, the optimal strategy for the hard power constraint problem consists of transmitting the innovation linearly, at each stage, at the maximum permissible power level. For the soft power constraint version the structure of the solution was found to be similar, with the optimal power levels being found via solving a nonlinear optimal control problem.

In Chapter 5 we generalized the results on decentralized, two-person teams obtained in Chapter 2, by allowing the action of one agent to be transmitted to the other agent through a number of noisy channels simultaneously, instead of being transmitted through a single noisy channel. We showed that if all channel noises are independent of the input variable, then linear strategies are optimal, even if the first agent observes a noise corrupted version of the input, the linear coefficients being found through a related parameter optimization problem. We further showed that when the channel noises are indeed correlated with the input variable, then there are instances in which the strategies which are optimal over the affine class may be outperformed by nonlinear strategies, even when the first agent observes an uncorrupted version of the input variable.

In Chapter 6, the results of Chapter 2 were generalized to cases with more than two decision makers. We considered problems with a single transmitting agent and multiple receiving agents, problems with multiple transmitting as well as receiving agents, and finally problems with multiple transmitting agents and a single receiving agent. For problems involving a single transmitting agent and multiple receiving agents, we showed that the optimal strategies are linear when either the first agent observes an uncorrupted version of the input, or when all channel noises are independent of the input. For problems

with multiple transmitting as well as receiving agents we again found that the optimal strategies are linear when either all transmitting agents observe an uncorrupted version of the variable to be transmitted or when all channel noises are independent of this variable. However, for the simplest classes of problems involving multiple transmitting agents and a single receiving agent, we found instances in which the strategies which are optimal over the affine class may be outperformed by appropriately chosen nonlinear strategies, even when the first agent observes an uncorrupted version of the input to be transmitted, and the channel noises are independent of the input. For this case, we provided strategies which are optimal within the affine class.

In Chapter 7, we allowed incomplete statistical description of the channel used to transmit measurements between the decentralized agents, and sought optimal solutions under a worst-case scenario. Assuming the unknown part of the channel noise to be controlled by an adversary or "jammer," we viewed the problems as zero-sum games. We considered a number of cases depending on whether there were hard power constraints or soft power constraints on the decision variables. The unknown channel noise was allowed to be correlated with either the input or the output of the encoder, i.e., the jammer was assumed to have the ability to tap the channel. We found that if the encoder is probabilistic and the jammer taps the input to the encoder, then saddle-point solutions exist under all fidelity criteria. If, however, the encoder structure is restricted to be deterministic, then saddle-point solutions do not exist, but both maximin and minimax strategies may be found. The maximin value here was found to be the same as the maximin value of a corresponding game where the encoder is allowed to be random, with the jammer still tapping the input to the encoder; and the minimax value here was found to be

be outperformed by appropriately chosen nonlinear strategies. We considered optimality over the affine class for problems involving a general j-th order model, and showed that within this class, the optimal strategy for the hard power constraint problem consists of transmitting the innovation linearly, at each stage, at the maximum permissible power level. For the soft power constraint version the structure of the solution was found to be similar, with the optimal power levels being found via solving a nonlinear optimal control problem.

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the same as the minimax value of a corresponding game where the jammer taps the output of the encoder.

The problems studied in this thesis can be viewed as important prototype problems, which could be considered essential building blocks for a general theory of multistage distributed decision making under nonclassical information, and possibly partial statistical description. A major focus has been on the question: Are all stochastic team problems which involve simultaneous communication and control, and hence exhibit nonclassical information patterns, inherently difficult and complex? We have shed some light on this question, obtained fundamental results for two-stage stochastic teams, and have made some contributions towards the development of a general theory for multistage (finite and infinite horizon) stochastic control and team problems with nonclassical information, where the control (decision) variable affects not only the state trajectory but also the quality of information available to the decision makers.

APPENDIX A

REFORMULATION OF THE TEAM PROBLEM

In this appendix we show how the squares may be completed to obtain the equivalent problems PS and PH from the problems PS° and PH°, respectively.

We first note that for $k>i$ we may write

$$x_k x_i = \sum_{m=0}^{i} \delta_{k,i,m} x_m^2 + \sum_{n=i}^{k-1} p_{k,i,n,i} v_n x_i + \sum_{n=0}^{i-1} \Big(\sum_{m=n-(j-2)}^{n} p_{k,i,n,m} v_n x_m \Big)$$

where the δ's are defined recursively as follows:

$$\delta_{100} = \rho_{10}$$

$$\delta_{m,0,0} = \rho_{m,0} + \sum_{k=1}^{m-1} \rho_{m,k} \delta_{k,0,0} \text{ for } 1<m\leqslant j-1$$

$$\delta_{k,m,p} = \sum_{n=m+1}^{k-1} \rho_{k,n} \delta_{n,m,p} + \sum_{n=p}^{m-1} \rho_{k,n} \delta_{m,n,p} \text{ for } p<m$$

$$\delta_{k,m,m} = \rho_{k,m} + \sum_{n=m+1}^{k-1} \rho_{k,n} \delta_{n,m,m}$$

and the p's are defined by

$$p_{k,m,k-1,m} = -1$$

$$p_{k,m,k-n,m} = \sum_{i=1}^{n-1} \rho_{k,k-i} p_{k-i,m,k-n,m} \text{ for } 2\leqslant n\leqslant k-m$$

$$p_{k,m,n,i} = \sum_{t=m+1}^{k-1} \rho_{k,t} p_{t,m,n,i} + \sum_{t=0}^{m-1} \rho_{k,t} p_{m,t,n,i} \,.$$

We define $p_{a,b,c,d}$ to be zero whenever any of the following conditions holds:

(i) $d < 0$

(ii) $d > \mathrm{Min}(b,c)$

(iii) $d = b$ and $c > k-1$

(iv) $c < b$ and $d < c - (j-2)$

and $\delta_{a,b,c}$ is defined to be zero whenever $c > b$.

We therefore have

$$(\bar{a}_i x_i + \bar{a}_{i-1} x_{i-1} + \ldots + \bar{a}_{i-(j-1)} x_{i-(j-1)})^2$$

$$= \sum_{k=0}^{i-1} v_k \Big(\sum_{m=k-(j-2)}^{k} 2r_{i+1,k,m} x_m \Big) + \sum_{k=0}^{i} q_{i+1,k} x_k^2$$

where the r's and q's are given as follows:

$$r_{i+1,n,p} = \sum_{k=i-(j-2)}^{i} \; \sum_{m=i-(j-1)}^{k-1} \bar{a}_k \bar{a}_m p_{k,m,n,p} ,$$

$$q_{i+1,n} = \sum_{k=i-(j-2)}^{i} \; \sum_{m=i-(j-1)}^{k-1} \bar{a}_k \bar{a}_m \delta_{k,m,n} \text{ for } 0 \leqslant n \leqslant i-j$$

and

$$q_{i+1,n} = a_n^2 + \sum_{k=i-(j-2)}^{i} \; \sum_{m=i-(j-1)}^{k-1} \bar{a}_k \bar{a}_m \delta_{k,m,n} \text{ for } i-j < n < i .$$

We can now write

$$\Big(\sum_{n=i-(j-1)}^{i} \rho_{i+1,n} x_n \Big)^2 = \sum_{k=0}^{i-1} v_k \Big(\sum_{m=k-(j-2)}^{k} 2s_{i+1,k,m} x_m \Big) + \sum_{m=0}^{i} t_{i+1,m} x_m^2$$

and

$$a_i(\sum_{n=i-(j-1)}^{i} b_{i,n}x_n)^2 = \sum_{k=0}^{i-1} v_k(\sum_{m=k-(j-2)}^{k} 2s'_{i+1,k,m}x_m) + \sum_{m=0}^{i} t'_{i+1,m}x_m^2 .$$

Here $s_{i+1,k,m}$ and $t_{i+1,n}$ are given by expressions identical to those for $r_{i+1,k,m}$ and $q_{i+1,n}$, respectively, with the $\bar{a}_n$ being replaced by $\rho_{i+1,n}$, and $s'_{i+1,k,m}$ and $t'_{i+1,n}$ are given similarly by replacing $\bar{a}_n$ by $b_{i,n}$ and further multiplying by a_i.

The completing of the squares now proceeds as follows. At the first step we have

$$\begin{aligned} &c_{N+1}x_{N+1} + d_N v_N^2 \\ &= d_N v_N^2 + c_{N+1}(v_N^2 - 2v_N(\sum_{m=N-(j-1)}^{N} \rho_{N+1,m}x_m) + m_N^2 \\ &+ \sum_{k=0}^{N-1} v_k(2\sum_{m=k-(j-2)}^{k} s_{N+1,k,m}x_m) + \sum_{m=0}^{N} t_{N+1,m}x_m^2) \end{aligned}$$

i.e., we can complete the first square to obtain

$$\begin{aligned} a_N &= d_N + c_{N+1} \\ b_{N,N} &= \frac{c_{N+1}\rho_{N+1,N}}{a_N} \\ &\vdots \\ b_{N,N-(j-1)} &= \frac{c_{N+1}\rho_{N+1,N-(j-1)}}{a_N} . \end{aligned}$$

Coming to the second step now, we have the following terms containing v_{N-1}:

$$\begin{aligned} &d_{N-1}v_{N-1}^2 + c_N x_N^2 + (t_{N+1,N} - t'_{N+1,N})x_N^2 \\ &\qquad + 2v_{N-1}(\sum_{m=N-1-(j-2)}^{N-1} (s_{N+1,N-1,m} - s'_{N+1,N-1,m})x_m . \end{aligned}$$

Letting

$$c'_N \equiv c_N + (t_{N+1,N} - t'_{N+1,N})$$

we obtain

$$a_{N-1} = d_{N-1} + c'_N$$

and expanding x_N^2 we get the following expressions for $b_{N-1,n}$'s:

$$b_{N-1,N-1} = \frac{c'_N \rho_{N,N-1} - (s_{N+1,N-1,N-1} - s'_{N+1,N-1,N-1})}{a_{N-1}}$$

$$\vdots$$

$$b_{N-1,N-j+1} = \frac{c'_N \rho_{N,N-j+1} - (s_{N+1,N-1,N-j+1} - s'_{N+1,N-1,N-j+1})}{a_{N-1}}$$

$$b_{N-1,N-j} = \frac{c'_N \rho_{N,N-j}}{a_{N-1}} .$$

Moving on to the third step, we have the following expression for terms containing v_{N-2}:

$$d_{N-2} v_{N-2}^2 + c_{N-1} x_{N-1}^2 + (t_{N+1,N-1} - t'_{N+1,N-1}) x_{N-1}^2 + (t_{N,N-1} - t'_{N,N-1}) x_{N-1}^2$$

$$+ 2 v_{N-2} (\sum_{m=N-2-(j-2)}^{N-2} (s_{N+1,N-2,m} - s'_{N+1,N-2,m}) x_m$$

$$+ 2 v_{N-2} (\sum_{m=N-2-(j-2)}^{N-2} (s_{N,N-2,m} - s'_{N,N-2,m}) x_m .$$

Letting

$$c'_{N-1} \equiv c_{N-1} + (t_{N+1,N-1} - t'_{N+1,N-1}) + (t_{N,N-1} - t'_{N,N-1})$$

we obtain

$$a_{N-2} = c'_{N-1} + d_{N-2}$$

$$b_{N-2,N-2} = \frac{c'_{N-1}\rho_{N-1,N-2} - (s_{N+1,N-2,N-2} - s'_{N+1,N-2,N-2}) - (s_{N,N-2,N-2} - s'_{N,N-2,N-2})}{a_{N-2}}$$

$$\vdots$$

$$b_{N-2,N-j} = \frac{c'_{N-1}\rho_{N-1,N-j} - (s_{N+1,N-2,N-j} - s'_{N+1,N-2,N-j}) - (s_{N,N-2,N-j} - s'_{N,N-2,N-j})}{a_{N-2}}$$

$$b_{N-2,N-j-1} = \frac{c'_{N-1}\rho_{N-1,N-j-1}}{a_{N-2}}\,.$$

Proceeding in a similar fashion we can obtain the general expression for the a's and b's as follows:

$$a_{k-1} = c'_k + d_{k-1}$$

where

$$c'_k = c_k + \sum_{n=k+1}^{N+1} (t_{n,k} - t'_{n,k})\,;$$

for $k-j+1 \leqslant m \leqslant k-1$ we have

$$b_{k-1,m} = \frac{c'_k\rho_{k,m} - \sum_{n=k+1}^{N+1} (s_{n,k-1,m} - s'_{n,k-1,m})}{a_{k-1}}$$

and

$$b_{k-1,k-j} = \frac{c'_k\rho_{k,k-j}}{a_{k-1}}\,.$$

This completes the first step of the transformation and we now have a cost functional of the form

$$E[\sum_{i=0}^{N} a_i(v_i - \sum_{k=0}^{j-1} b_{i,i-k}x_{i-k})^2] + \bar{c}_N$$

where $\bar{c}_N$ is a constant given by

$$\bar{c}_N = \sum_{i=0}^{N} c'_{i+1}\sigma^2_{m_i} + c'_0\sigma^2_{m_0}.$$

To complete the transformation, we now need a redefinition of the v's. Towards this end, we first note that with $\tilde{x}$'s defined by

$$\tilde{x}_{i+1} = \sum_{k=0}^{j-1} \rho_{i+1,i-k}\tilde{x}_{i-k} + m_i ,$$

we have for i=0,1,2,...

$$x_{i+1} = \tilde{x}_{i+1} - \bar{v}_i$$

where

$$\bar{v}_i = v_i + \sum_{k=0}^{j-1} \rho_{i+1,i-k}\bar{v}_{i-(k+1)} .$$

Therefore, we have

$$\begin{aligned}
&v_i - \sum_{k=0}^{j-1} b_{i,i-k}x_{i-k} \\
&= v_i - \sum_{k=0}^{j-1} b_{i,i-k}(\tilde{x}_{i-k} - \bar{v}_{i-k-1}) \\
&= v_i + \sum_{k=0}^{j-1} b_{i,i-k}\bar{v}_{i-k-1} - \sum_{k=0}^{j-1} b_{i,i-k}\tilde{x}_{i-k} \\
&= \sum_{k=0}^{j-1} b_{i,i-k}\tilde{x}_{i-k}
\end{aligned}$$

and hence, the new $\tilde{v}$'s are defined in terms of the original v's by

$$\tilde{v}_i = v_i + \sum_{k=0}^{j-1} b_{i,i-k}\bar{v}_{i-k-1} ,$$

which is an invertible transformation. □

APPENDIX B

THE GAUSSIAN CHANNEL WITH SIDE INFORMATION AT THE DECODER

Consider Problem P1 defined below where the objective is to simultaneously design the measurement policy γ and the control policy δ to minimize a quadratic cost.

Problem P1

$$\underset{\gamma,\delta}{\text{Minimize}}\ J(u,v) = E[k_0 u^2 + (v-x)^2] \tag{B.1}$$

where

$$u = \gamma(x) \tag{B.2a}$$

$$v = \delta(x_1, x_2) \tag{B.2b}$$

and

$$x_1 = x + w_1 \tag{B.3a}$$

$$x_2 = u + w_2\,. \tag{B.3b}$$

Here x, w_1 and w_2 are zero mean, independent, Gaussian random variables, with variances σ_x^2, $\sigma_{w_1}^2$ and $\sigma_{w_2}^2$, respectively.

To facilitate the study of Problem P1, we formulate a related problem, P2, which is the hard constraint version of Problem P1, i.e., the term $E[k_0 u^2]$ is removed from the cost functional and the further stipulation is made that

$$E[\gamma^2(x)] \leqslant P^2\,. \tag{B.4}$$

We therefore have Problem P2 below.

Problem P2

$$\underset{\gamma,\delta}{\text{Minimize}}\ J_2(u,v) = E[(v-x)^2] \tag{B.5}$$

given (2), (3) and (4).

Problem P2 is represented schematically in Figure B.1 below. Note that this problem may be viewed as one of designing encoding and decoding strategies in the presence of side information at the decoder. If the side information were absent, then this would be a special case of the problem studied in Bansal and Başar [1987a], without a product term between the decision variables, and the linear solution would be optimal. For the above problem, however, the linear solution is not optimal as we shall elucidate in the sequel.

Encoding-decoding problems with side information have been the subject of some previous investigations. The rate distortion for source coding with side information at the decoder has been studied for general sources in Wyner [1978]. It has been shown that if the source and side information are jointly Gaussian, then the minimum rate (in the usual Shannon sense) required for encoding the source at a distortion level about d, is equal to

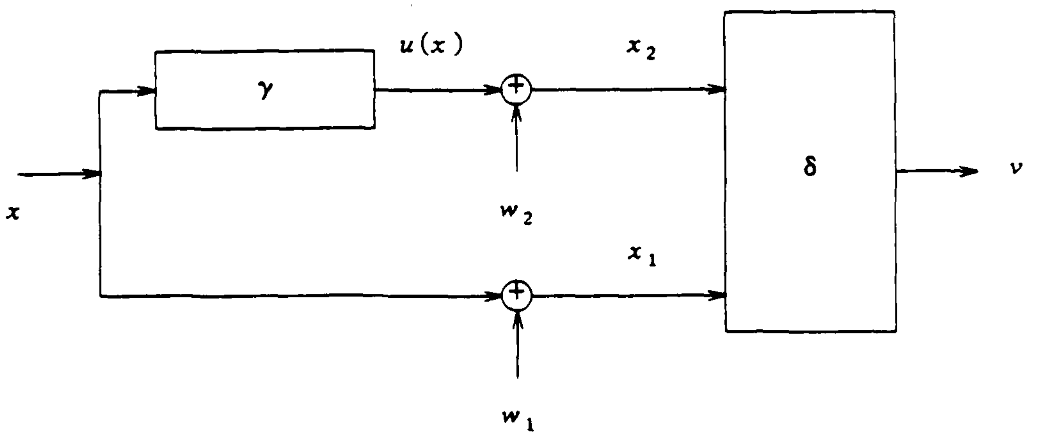

Figure B.1. Schematics of Problem P2.

the rate which would be required if the encoder (as well as the decoder) had access to this side information. Since the coding theorem uses the fact that long ergodic sequences are asymptotically typical and therefore can be encoded into sequences that have the distribution which achieves capacity, Wyner's result implies that if block encoding and arbitrary delays were permissible, then the least mean square error for Problem P2 would be the same as that for the system depicted in Figure B.2 below. Note that the only difference between Figures B.1 and B.2 is that in the latter the encoder too has access to the side information.

The situations we envisage are control theoretic applications, where the encoder output represents system measurements and the decoder has to control the system in real time. Thus, bounds which are asymptotically tight are of little use in our application, where sample by sample transmission is required. Real-time coding-decoding problems have also been studied earlier, notably in Walrand and Varaiya [1983], where *finite* sources are considered and feedback information has been shown to be useful for the

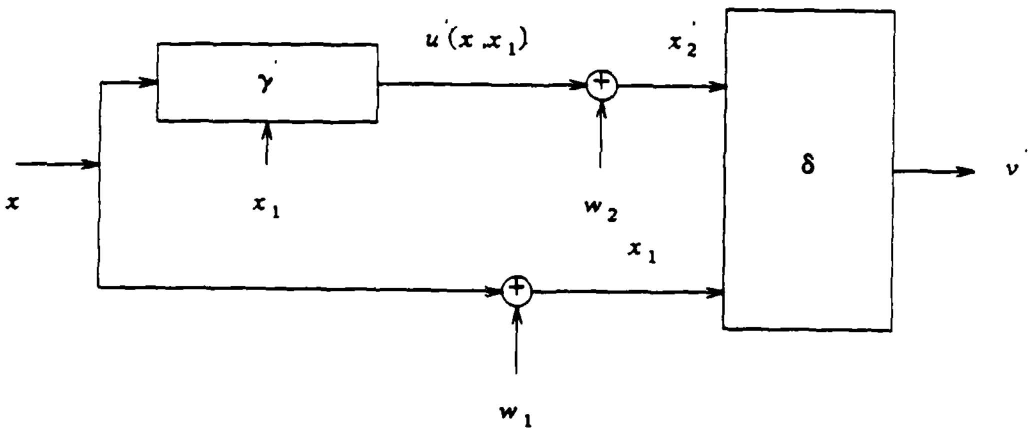

Figure B.2. The encoder has side information also.

causal encoding-decoding problem. In the problems of interest to us here, feedback is not permitted.

Wyner's [1978] result implies that if block encoding were permitted, then the least mean square error for Problem P2 would be the same as in the case illustrated by Figure B.2. For the situation depicted in Figure B.2 we have

$$I(x;x_1) = \frac{1}{2}\log\left(\frac{\sigma_x^2+\sigma_{w_1}^2}{\sigma_{w_1}^2}\right) \tag{B.6}$$

and

$$I(x;x'_2\,|\,x_1) \leqslant \frac{1}{2}\log\left(\frac{P^2+\sigma_{w_2}^2}{\sigma_{w_2}^2}\right). \tag{B.7}$$

Now since

$$I(x;x_1,x'_2) = I(x;x_1) + I(x;x'_2\,|\,x_1), \tag{B.8}$$

we have

$$I(x;x_1,x'_2) \leqslant \frac{1}{2}\log\left(\frac{(\sigma_x^2+\sigma_{w_1}^2)(P^2+\sigma_{w_2}^2)}{\sigma_{w_1}^2\sigma_{w_2}^2}\right). \tag{B.9}$$

The rate distortion function for a Gaussian source of variance σ_x^2 under a mean square distortion criterion is

$$R(d) = \mathrm{Max}\left(0,\frac{1}{2}\log\frac{\sigma_x^2}{d}\right). \tag{B.10}$$

Further,

$$I(x;x_1,x'_2) \geqslant I(x;v') \geqslant \frac{1}{2}\log\frac{\sigma_x^2}{E((x-v')^2)} \tag{B.11}$$

which implies that

$$E[(x-v')^2] \geqslant \frac{\sigma_x^2\sigma_{w_1}^2\sigma_{w_2}^2}{(\sigma_x^2+\sigma_{w_1}^2)(P^2+\sigma_{w_2}^2)} \triangleq \bar{d}\,. \tag{B.12}$$

Now if we use the causal policy

$$u(x) = \lambda'(x - E(x \mid x_1)) \tag{B.13}$$

where λ' satisfies

$$(\lambda')^2 = \frac{P^2(\sigma_x^2+\sigma_{w_1}^2)}{(\sigma_x^2\sigma_{w_1}^2)} \tag{B.14}$$

then this least possible distortion $\bar{d}$ is attained with equality, implying that the linear strategy indicated is optimal for the problem with feedback.

In Problem P2 feedback is not available, and hence the optimal linear strategy consists of using

$$u(x) = \lambda x \tag{B.15a}$$

$$v = E[x \mid x_1,x_2] \tag{B.15b}$$

which yields the mean square distortion d^* given by

$$d^* = \frac{\sigma_x^2\sigma_{w_1}^2\sigma_{w_2}^2}{P^2\sigma_{w_1}^2+\sigma_{w_2}^2(\sigma_x^2+\sigma_{w_1}^2)}\,. \tag{B.16}$$

Clearly $d^* > \bar{d}$, since $P^2\sigma_x^2 > 0$. This is not unexpected since in the case of $\bar{d}$, additional information was used at the encoder. As discussed earlier, if arbitrarily large delays are allowed, then the least distortion for P2 is also $\bar{d}$, hence strategies may be constructed via block encoding which would improve upon the case of linear sample by sample transmission.

The important issue now is whether a reduction in the least mean square error d^* (obtained via linear coding) is possible *without* block encoding. We now show that the optimal causal linear strategy may be outperformed by a causal nonlinear strategy, i.e., without block encoding and thus with no delay.

We first provide an example where the optimum linear strategy for Problem P2 may be outperformed by an appropriately chosen nonlinear policy. Towards this end, we propose the following design:

$$u = x + \epsilon \operatorname{sgn} x \tag{B.17}$$

$$v = \begin{cases} (x_1+x_2-\epsilon)/2 & \text{if } x_1 \geqslant 0 \\ (x_1+x_2+\epsilon)/2 & \text{if } x_1 < 0 \end{cases} . \tag{B.18}$$

We next find the mean square error using the policy proposed in (B.17) and (B.18).

We have

$$\begin{aligned} E[(x-v)^2] &= E[(x-v)^2 \mid x_1 \geqslant 0]\text{Prob}[x_1 \geqslant 0] + E[(x-v)^2 \mid x_1 < 0]\text{Prob}[x_1 < 0] \\ &= E[(x-v)^2 \mid x_1 \geqslant 0] . \end{aligned}$$

Now,

$$E[(x-v)^2 \mid x_1 \geqslant 0]$$

$$= E[(x-v)^2 \mid x_1 \geqslant 0, x \geqslant 0] \mathrm{Prob}[x \geqslant 0 \mid x_1 \geqslant 0]$$

$$+ E[(x-v)^2 \mid x_1 \geqslant 0, x < 0] \mathrm{Prob}[x < 0 \mid x_1 \geqslant 0] .$$

Since

$$E[(x-v)^2 \mid x_1 \geqslant 0, x \geqslant 0] = \frac{\sigma_{w_1}^2 + \sigma_{w_2}^2}{4}$$

and

$$E(x-v)^2 \mid x_1 \geqslant 0, x < 0] = \frac{\sigma_{w_1}^2 + \sigma_{w_2}^2}{4} + \epsilon^2 ,$$

we have

$$E[(x-v)^2 \mid x_1 \geqslant 0] = \frac{\sigma_{w_1}^2 + \sigma_{w_2}^2}{4} + \epsilon^2 \mathrm{Prob}[x < 0 \mid x_1 \geqslant 0] .$$

Now $\mathrm{Prob}(x < 0 \mid x_1 \geqslant 0)$ may be easily computed, since the joint density of x and x_1 is known:

$$p_{x,x_1}(\cdot,\cdot) = \frac{1}{2\pi \sigma_x \sigma_{w_1}} \exp\left(- \frac{(x_1 - x)^2}{2\sigma_{w_1}^2} - \frac{x^2}{2\sigma_x^2} \right) .$$

Further,

$$\mathrm{Prob}(x < 0 \mid x_1 \geqslant 0) = 1 - \mathrm{Prob}(x > 0 \mid x_1 \geqslant 0)$$

$$\mathrm{Prob}(x \geqslant 0 \mid x_1 \geqslant 0) = \frac{\mathrm{Prob}(x \geqslant 0, x_1 \geqslant 0)}{\mathrm{Prob}(x_1 \geqslant 0)} \triangleq \frac{I}{1/2} = 2I$$

where

$$I = \int_0^\infty \int_0^\infty p_{x,x_1} dx\, dx_1 .$$

Letting $\sigma_{w_1}^2 = 1$ we get

$$I = \int_0^\infty \frac{1}{2\pi\sigma_x} \exp(-\frac{x^2}{2\sigma_x^2})dx \int_0^\infty \exp(-\frac{(y-x)^2}{2})dy$$

$$= \int_0^\infty \frac{1}{\sqrt{2\pi}\sigma_x} \exp(\frac{-x^2}{2\sigma_x^2})dx \frac{1}{\sqrt{2\pi}} \int_{-x}^\infty e^{-t^2/2}dt$$

$$= \int_0^\infty \frac{1}{\sqrt{2\pi}\sigma_x} \exp(\frac{-x^2}{2\sigma_x^2})\phi(-x)dx$$

$$= \int_0^\infty \frac{1}{\sqrt{2\pi}} \exp(\frac{-z^2}{2})\phi(-\sigma_x z)dz .$$

Thus, I may be evaluated and the error using nonlinear policies:

$$C_{NL} \triangleq \frac{1+\sigma_{w_2}^2}{4} + \epsilon^2(1-2I)$$

may be found, and hence the mean square error using linear strategies and that using this nonlinear strategy may now be compared.

We now provide some examples where the optimum linear strategy for Problem P2 is outperformed by a nonlinear strategy of the form given by (B.17) and (B.18). We then illustrate how these results may be used to construct instances of Problem P1 where a nonlinear strategy may be used to outperform the optimal linear strategy.

Consider Problem P2 with parameter values $\sigma_x^2 = 100.0$, $\sigma_{w_1}^2 = \sigma_{w_2}^2 = 1.0$. Letting $P^2 = 85.0423$, we get $\epsilon = -1$ and the nonlinear strategy achieves a mean square error of 0.53172. The optimum linear strategy at the above power level attains a mean square error of 0.53751, which is inferior to that attained by the nonlinear policy.

Table B.1 below gives other instances of Problem P2 where the nonlinear policy outperforms the optimum linear policy, the third column indicating the improvement $(C_L - C_{NL})$, i.e., the difference between the mean square errors for the two cases, and the last column indicating percentage improvement over the optimum linear design. In all the cases considered in Table B.1, we have used $\sigma_{w_1}^2 = \sigma_{w_2}^2 = 1.0$.

TABLE B.1. SOME INSTANCES OF PROBLEM P2 WHERE NONLINEAR STRATEGIES OUTPERFORM OPTIMUM LINEAR STRATEGIES.

σ_x	$-\epsilon$	Improvement	% change
4.0	1.00	0.001375	0.24
6.0	1.10	0.001736	0.31
6.0	0.90	0.009627	1.74
6.0	0.80	0.011973	2.19
6.0	0.70	0.013262	2.46
6.0	0.50	0.012691	2.41
10.0	0.80	0.009148	1.73
10.0	0.60	0.009994	1.92
10.0	0.55	0.009813	1.89

We now construct an instance of Problem P1 by finding k_0 which corresponds to the appropriate hard power constraint. Returning to the example with $\sigma_x^2 = 100.0$, $\sigma_{w_1}^2 = \sigma_{w_2}^2 = 1.0$, we note that when the policies are linear, the optimum power level is given by

$$P^{*2} = \arg\min_{P^2 \geq 0} \left[k_0 P^2 + \frac{\sigma_x^2 \sigma_{w_1}^2 \sigma_{w_2}^2}{P^2 \sigma_{w_1}^2 + \sigma_{w_2}^2 (\sigma_x^2 + \sigma_{w_1}^2)} \right]$$

which implies that k_0 and P^{*2} satisfy

$$k_0 - \frac{100}{(P^{*2}+101)^2} = 0,$$

and for the case in consideration with $P^2 = 85.0423$, we may let

$$k_0 = \frac{100}{(186.0423)^2} = 0.0028892\ .$$

Therefore, for Problem P1 with parameter values $k_0 = 0.0028892$, $\sigma_x^2 = 100.0$, $\sigma_{w_1}^2 = \sigma_{w_2}^2 = 1.0$, the optimum cost over the linear class is given by $(0.0028892 \times 85.0423 + 0.53751 =)\ 0.78321$ whereas a nonlinear policy of the form (B.17) and (B.18) with $\epsilon = -1$ attains 0.77742.

$\square$

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